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In Hwang's report he presented the Bayesian decision theoretic approach in which the problem is related to a stopping problem involving Brownian motion, a stopping cost represented by a single cost parameter c, and an initial point (y, s) depending on the prior normal distribution of an unknown parameter and other known parameters of the problem. The solution of the problem is represented by dividing up the set $\{(y,s) : s > 0\}$ into a continuation set C and a stopping set S.

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HARVARD UNIVERSITY

DEPARTMENT OF STATISTICS

TEL. 617-495-5496
FAX. 617-496-8057

SCIENCE CENTER
ONE OXFORD STREET
CAMBRIDGE, MASSACHUSETTS 02138

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Bounds and Asymptotic Expansions for Solutions of the Free Boundary Problems Related to Sequential Decision Versions of a Bioequivalence Problem

John Bather
University of Sussex

and

Herman Chernoff
Harvard University

Technical Report No. ONR-C-15

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1. Introduction

The numerical solutions of two related variations of a sequential version of a form of the bioequivalence problem was presented in a report by Hwang (1991). In that report he referred to our unpublished results bounding the solutions and providing asymptotic expansions. The present report has two major functions. One is to derive and amplify these results, and incidentally to correct an error. The second is to gather together in one place, and with relatively little of the abbreviation characteristic of previous publications, many of the details that are useful in deriving the asymptotic expansions, e.g., see Breakwell and Chernoff (1964), Chernoff (1965a, 1972), and Chernoff and Petkau (1981).

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In Section 2, we shall describe the formulations of the bioequivalence problems, the corresponding stopping problems and the relevance of the free boundary problem. In Section 3 we describe the results of Hwang and how our theoretical results related to these. In Section 4, we cumulate details about separable solutions of the heat equation. These will serve as a repository for future reference and many of these will be relatively unnecessary for this particular problem.

In Section 5 we will derive the bounds and in Section 6, the asymptotic expansions will be calculated formally. No attempt will be made to repeat the type of proof of Chernoff (1965a) that these expansions are true asymptotic expansions.

For these particular problems, Hwang's computer program, modeled on Chernoff and Petkau (1984) provides a very good numerical solution, and the bounds and expansions are of little more than academic interest. Nevertheless these theoretical results were derived first, and provided a qualitative view of the solution which seemed quite strange at the time. Without that view, it may have been difficult to produce a good computation. In one particular case, Hwang shows when the accuracy of the numerical procedure has to be improved, as it readily can with more computer time, in order to match better with some of the asymptotic expansions.

Most of the bounds in Section 5 apply relatively simple principles compared to those pioneered by Bather (1962, 1970) and illustrated in Chernoff (1972). In spite of the length of this manuscript, the amount of time the authors spent in their collaboration leading to these theoretical results was relatively small, because it was felt that enough was known to facilitate the numerical calculation. However a miscalculation at the time failed to indicate the presence of another term of theoretical interest, which is still unaccounted for and which will be discussed in Section 6.

2. Problem Formulation and Summary of Results

There are several alternative formulations of the bioequivalence problem. Some time ago Bather, Chernoff and Petkau (1988) presented the following formulations. In his description of the formulation, Hwang referred to work by Metzler (1974), O'Quigley and Baudoin (1988) and Selwyn et. al. (1981).

Let X be a normally distributed variable, with unknown mean μ and known variance σ^2, representing the difference between the outcomes of two treatments on a matched pair of subjects. One of these treatments is the standard and the other is a new treatment which is hoped to have matching pharmacological properties. As observations on X cumulate, the decision is made to continue or stop sampling. When sampling is stopped, the decision is made to declare difference or equivalence. Presumably if difference is declared, the search for a new equivalent formulation is repeated.

The costs involved are that of having to start over if nonequivalence is announced, that of having to accept the mean μ, which may be different from 0, when equivalence is announced, and a cost of sampling. The cost of starting over is some constant k_0 independent of μ since μ is simply a characteristic of the candidate drug which is discarded. The cost of accepting bioequivalence should depend on how large μ is. We will consider two versions. In the first this cost is $k_1\mu^2$ and in the second it is $k_2|\mu|$. Finally, we shall assume that the cost of sampling is linear in the sample size n, i.e., $c_0 n$

for n pairs observed. Implicit in this sequential version of the problem is the assumption of immediate observed response to treatment.

Clearly the operating characteristics of any procedure applied in this problem will depend on the unknown mean μ. The concept of the optimal procedure requires some criterion of optimality. By accepting a Bayesian formulation with a prior normal distribution with mean μ_0 and variance σ_0^2, we establish a clear cut optimization problem, which can in principle be solved directly by the backward induction of dynamic programming.

The theoretical study of this problem can be facilitated by converting the problem to a related one where the sum of the first n i.i.d. observations $X_1, X_2, \ldots, X_n$ is replaced by Brownian Motion $X(t)$ with unknown drift μ and known variance σ^2 per unit time. The original discrete time problem may be regarded as a special version of the continuous time problem where stopping is permitted only at certain discrete times. One outcome of this continuous time formulation is that the class of problems can be reduced to a one parameter family of problems, the numerical solution of which can be approximated arbitrarily well by techniques outlined in Chernoff and Petkau (1986). Another outcome is the relation of the optimization problem to the solution of a free boundary problem (FBP) involving the heat equation.

In the discrete time Bayesian version, the posterior probability distribution of μ after n observations is given by

$$\mathcal{L}(\mu | X_1, X_2, \ldots, X_n) = N(Y_n, s_n) \tag{2.1}$$

where

$$Y_n = \frac{\mu_0 \sigma_0^{-2} + (X_1 + \ldots + X_n)\sigma^{-2}}{\sigma_0^{-2} + n\sigma^{-2}} \tag{2.2}$$

and

$$s_n = (\sigma_0^{-2} + n\sigma^{-2})^{-1} . \tag{2.3}$$

In the continuous time version

$$\mathcal{L}(\mu | X(t'), 0 \le t' \le t) = N(Y(s), s) \tag{2.4}$$

where

$$Y(s) = (\mu_0 \sigma_0^{-2} + X(t)\sigma^{-2})/(\sigma_0^{-2} + t\sigma^{-2}) \tag{2.5}$$

and

(2.6) $$s = (\sigma_0^{-2} + t\sigma^{-2})^{-1} .$$

Moreover, it can be shown that $Y(s)$ is a Brownian motion with zero drift in the $-s$ scale originating at $(y_0, s_0) = (\mu_0, \sigma_0^2)$, i.e.,

(2.7) $$E[dY(s)] = 0$$

(2.8) $$\mathrm{Var}[dY(s)] = -ds$$

Note that as t increases from 0 to ∞, s decreases from s_0 to 0.

The posterior risk in stopping at $Y(s) = y$, is given in the first version, which we shall refer to as Problem 1, by

$$\min(k_0, E[\mu^2|Y(s) = y]) = \min(k_0, k_1(y^2 + s)).$$

In the second version or Problem 2, the posterior risk can be calculated to be

$$\min(k_0, E[|\mu||Y(s) = y]) = \min(k_0, k_2 s^{1/2} G_{1e}(ys^{-1/2}))$$

where

(2.9) $$G_{1e}(\alpha) = 2\{\phi(\alpha) + \alpha[\Phi(\alpha) - 1/2]\}$$

and ϕ and Φ are the standard normal density and cumulative. Thus the stopping risk in Problem 1 is given by

(2.10) $$\begin{aligned} d_{11}(y,s) &= c_0 t + \min(k_0, k_1(y^2 + s)) \\ &= c_0\sigma^2 s^{-1} + \min(k_0, k_1(y^2+s)) - c_0\sigma^2\sigma_0^{-2} \end{aligned}$$

and in Problem 2 by

(2.11) $$d_{21}(y,s) = c_0\sigma^2 s^{-1} + \min(k_0, k_2 s^{1/2} G_{1e}(ys^{-1/2})) - c_0\sigma^2\sigma_0^{-2}.$$

For a stopping cost $d(y,s)$ there is a corresponding optimal Bayes Risk $\rho(y,s)$, given that we have reached $Y(s) = y$, and proceed optimally thereafter. But then it pays to stop if $\rho(y,s) = d(y,s)$ and to continue if $\rho(y,s) < d(y,s)$. But this means that the optimal stopping time $\tilde{S}$ corresponds to a procedure which stops when $(Y(s), s)$ reaches a stopping set $\tilde{\mathcal{S}} = \{(y,s) : \rho(y,s) = d(y,s)\}$. That is, the optimal procedure is one of a class of procedures which is defined by a set $\mathcal{S}$ and its complement $\mathcal{C} = \mathcal{S}^c$.

Any stopping time S, optimal or suboptimal, which is defined by a subset $\mathcal{S}$ of $\{(y,s) : s > 0\}$, has a corresponding Bayes risk given by

$$b(y,s) = E\{d(Y(S), S) | Y(s) = y\} \tag{2.12}$$

which satisfies the ordinary boundary problem

$$\frac{1}{2} b_{yy}(y,s) = b_s(y,s) \qquad \text{for } (y,s) \in \mathcal{C} \tag{2.13}$$

and

$$b(y,s) = d(y,s) \qquad \text{for } (y,s) \in \mathcal{S}\ . \tag{2.14}$$

The extra condition that determines the optimality of S is

$$b_y(y,s) = d_y(y,s) \qquad \text{for } (y,s) \in \partial\mathcal{S} \tag{2.15}$$

where $\partial\mathcal{S}$ is the boundary of $\mathcal{S}$. The pair $(\mathcal{S}, b)$ which satisfy (2.12) - (2.14) represents the solution of the *free boundary problem* (FBP) and relatively mild sufficiency conditions under which the solution of the FBP represents the solution of the optimum stopping problem, are discussed in Chernoff (1972). To paraphrase those conditions, they state that if the stopping rule determined by the solution of the FBP can't be improved trivially, the stopping rule is optimal.

Since the stopping set $\tilde{\mathcal{S}}$ does not depend on the initial value of (y,s), i.e., (μ_0, σ_0^2), we see that the set $\tilde{\mathcal{S}}$ represents a solution which applies simultaneously for all (μ_0, σ_0^2). The remaining parameters c_0, k_0, k_1 in Problem 1 can be incorporated into one essential parameter by use of a transformation of the form

$$Y^* = aY, \qquad s^* = a^2 s \tag{2.15}$$

which leaves $E[dY^*(s^*)] = 0$ and $\text{Var}[dY^*(s^*)] = -ds^*$, and by noting that the solution $\tilde{\mathcal{S}}$ of the FBP is not altered when d is multiplied by a constant or modified by the addition of a solution of the heat equation.

Now let $k_1/a^2k_0 = 1$, i.e.,

(2.17) $$a = (k_1/k_0)^{1/2} .$$

Then

(2.18) $$d_{11}(y,s) = k_0 d_{12}(y^*, s^*) - c_0\sigma^2\sigma_0^{-2}$$

where

(2.19) $$d_{12}(y,s) = c_1 s^{-1} + \min(1, y^2 + s)$$

and

(2.20) $$c_1 = c_0\sigma^2 k_1/k_0^2$$

and the solution of the FBP or optimization problem for d_{12} can be converted simply to that for d_{11}. Note that we now have a FBP with only one parameter c_1 to replace $\mu_0, \sigma_0^2, c_0, k_0$ and k_1.

A similar transformation works for Problem 2. Here we have, with $a = k_2/k_0$

(2.21) $$d_{21}(y,s) = k_0 d_{22}(y^*, s^*) - c_0\sigma^2\sigma_0^{-2}$$

where

(2.22) $$d_{22}(y,s) = c_2 s^{-1} + \min(1, s^{1/2} G_{1e}(ys^{-1/2}))$$

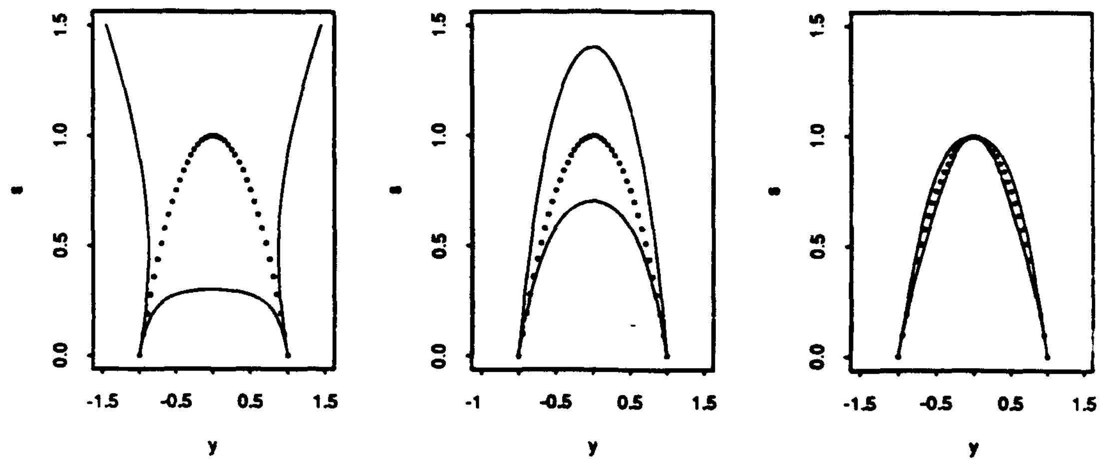
and

(2.23) $$c_2 = c_0\sigma^2 k_2^2/k_0^3$$

Later we shall see computational advantages for deriving asymptotic expansions by subtracting special solutions of the heat equation from d_{12} and d_{22}.

3. Character of the Solutions

Because of the symmetry of the loss functions as functions of μ, $\tilde{S}$, the optimal $\mathcal{S}$ and $\tilde{S}^c = \tilde{\mathcal{C}}$ are symmetric in y. These sets are adequately described by two curves, an inner and an outer curve given by $\tilde{y}_i(s)$ and $\tilde{y}_o(s)$ respectively as indicated in Figure 3.1 for 3 values of c_1. The inner curve bounds $\tilde{S}_i$, the subset of $\tilde{S}$ on which one declares bioequivalence and the outer curve bounds $\tilde{S}_0$, the remaining part of $\tilde{S}$ which leads to the declaration of nonequivalence. In between is the continuation set $\tilde{C}$.



c small $\quad$ c medium $\quad$ c large

Figure 1. A schematic representation of the optimal boundaries of Problem 1, for various values of c.

Since $\tilde{S}$ is monotone increasing in c there are several critical values, depending on which problem we consider, where $\tilde{S}$ changes character somewhat. For $c < \tilde{c}_o$ the outer curve does not close and $(0, s) \in \tilde{C}$ for arbitrary large values of s. If we let $\tilde{s}_i$ and $\tilde{s}_o$ be the values of s where $\tilde{y}_i$ and $\tilde{y}_o$ are zero, then we define $\tilde{c}_i$ as the infinimum of those values of c for which $\tilde{s}_i = \tilde{s}_o$. In Problem 1 all points of $y^2 + s = 1$ with the possible exception of (0,1) are in $\tilde{C}$. In Problem 2 the same holds for the points where $s^{1/2}G_{1e}(ys^{-1/2}) = 1$, with the possible exception of $(0, \pi/2)$.

Hwang tabulated $\tilde{s}_i$ and $\tilde{s}_o$ as functions of c and calculated $\tilde{c}_o$ and $\tilde{c}_i$. He found, for Problem 1, $\tilde{c}_{1i} \approx 0.75$ and $\tilde{c}_{1o} \approx 0.057$. Our bounds indicate that for $0 \le c_1 \le 1$, $\tilde{s}_{1i} \ge c_1^{1/2}$, and for $1/4 < c_1 \le 1$, $\tilde{s}_{1o} \le c_1/(2c_1^{1/2} - 1)$. These imply $\tilde{c}_{1i} \le 1$ and $\tilde{c}_{1o} \le 1/4$. Other bounds yield $\tilde{c}_{1i} \ge (2/\pi e)^{1/2} = 0.484$ and $\tilde{c}_{1o} \ge 0.0554$. Two of these four bounds are poor, and one is good.

Similar results apply for Problem 2 where Hwang calculated $\tilde{c}_{2i} \approx 0.75$ and $\tilde{c}_{2o} \approx 0.0518$. We derive $\tilde{s}_{2i} \geq (c_2/\phi_0)^{2/3}$ for $0 < c_2 \leq \pi/4$ and $\tilde{s}_{2o} \leq c_2/(3c_2^{1/3}\phi_0^{2/3} - 1)$ for $2\pi/27 < c_2 \leq \pi/4$, where $\varphi_0 = (2\pi)^{-1/2}$. We also find $(\pi/8e)^{1/2} = 0.380 \leq \tilde{c}_{2i} \leq \pi/4 = 0.785$ and $0.0498 \leq \tilde{c}_{2o} \leq 2\pi/27 = 0.233$.

When c becomes large the set $\tilde{C}$ shrinks but never completely vanishes. However, in the discrete time version of the problem, the effective $\tilde{C}$ shrinks still further and for sufficiently large cost of sampling it may pay not to take observations no matter what the normal prior distribution is. In particular the region near the points $(y, s) = (\pm 1, 0)$ representing very large values of t, and hence of large observed sample size, disappear from $\tilde{C}$ in the discrete time version of the problem.

This is borne out both, by Hwang's coarse approximations to the solutions of the free boundary problem, and his illustration of the discrete time correction based on the solution of the continuous time version.

4. Solutions of the Heat Equation

In this section we will concentrate mainly on separable solutions of the heat equation which are of the form $u = s^{n/2}H_n(ys^{-1/2})$. For these solutions H_n can be expressed in terms of confluent hypergeometric functions which can be represented in various forms. In particular we will present power series expansions for odd and even solutions. Also, for nonnegative integers n, we present representations for $H_n(\alpha)$ of the form

$$G_n(\alpha) = P_n(\alpha)\phi(\alpha) + Q_n(\alpha)\Phi(\alpha) \tag{4.1}$$

where $\alpha = ys^{-1/2}$, and $P_n(\alpha)$ and $Q_n(\alpha)$ are polynomials in α For negative integers n we will express the $H_n(\alpha)$ in terms of $\phi(\alpha)$ and

$$J(\alpha) = e^{-\alpha^2/2} \int_0^\alpha e^{x^2/2} dx \ . \tag{4.2}$$

Finally we shall present a representation appearing in Goursat (1942), and remark on a form of the solution useful for asymptotic expansions for large s.

In general, because of the relation between the heat equation and Brownian motion, one form of solution of the heat equation can be derived by assigning a "heat source" $h(y)$

to $\mathcal{S} = \{(y,s) : s = 0\}$. Then we have

$$u(y,s) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{s}} \phi\left(\frac{w-y}{\sqrt{s}}\right) h(w) dw = E h(y + W\sqrt{s}) , \tag{4.3}$$

where $\mathcal{L}(W) = N(0,1)$, provided h has no serious singularities and does not grow too rapidly as $z \to \infty$. We shall later examine this form when $h(z) = z^n$ and $|z^n|$ for nonnegative integer n.

If

$$u(y,s) = s^{n/2} H_n(\alpha) , \tag{4.4}$$

is a solution of the heat equation $u_{yy}/2 = u_s$, then

$$H_n''(\alpha) + \alpha H_n'(\alpha) = n H_n(\alpha) \tag{4.5}$$

Assuming $H_n(\alpha)$ is an even function of α with the expansion

$$H_{ne}(\alpha) = \sum_{j=0}^{\infty} c_{n,j} \alpha^{2j} \tag{4.6}$$

with $c_{n0} = 1$, we have, equating coefficients in (4.5)

$$c_{n,j} = \frac{n - 2j + 2}{(2j)(2j-1)} c_{n,j-1} = -\frac{1}{2} \frac{[-\frac{n}{2} + (j-1)]}{j[\frac{1}{2} + (j-1)]} c_{n,j-1}$$

and thus

$$H_{ne}(\alpha) = F\left(-\frac{n}{2}, \frac{1}{2}; -\frac{\alpha^2}{2}\right) \tag{4.7}$$

where

$$F(\beta, \gamma; x) = 1 + \frac{\beta}{\gamma} \frac{x}{1!} + \frac{\beta(\beta+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \ldots + \frac{\beta(\beta+1)\ldots(\beta+j-1)}{\gamma(\gamma+1)\ldots(\gamma+j-1)} \frac{x^j}{j!} + \ldots \tag{4.8}$$

Similarly for the odd function of α with expansion

$$H_{no}(\alpha) = \sum_{j=0}^{\infty} c_{n,j}^* \alpha^{2j+1} \tag{4.9}$$

with $c_{n0}^* = 1$, we have

$$c_{n,j}^* = -\frac{1}{2} \frac{[-(\frac{n-1}{2}) + (j-1)]}{j[\frac{3}{2} + (j-1)]} c_{n,j-1}^* \tag{4.10}$$

and

$$(4.11) \qquad H_{no}(\alpha) = \alpha F\left(-\frac{n-1}{2}, \frac{3}{2}; -\frac{\alpha^2}{2}\right)$$

In Tables 1 and 2, we list the coefficients for various integer values of n and j. In these expansions there is no need for n to be an integer. Incidentally, it is easy to see that the derivative of a solution of (4.5) is also a solution for a lower value of n. That is

$$H_n^{(3)}(\alpha) + \alpha H_n''(\alpha)(4.27) = (n-1)H_n'(\alpha).$$

It should be noted that if $H_n(\alpha)$ is even then $H_n'(\alpha)$ is odd and vice versa.

We now introduce an alternative view of these separable solution. Let $h(y) = y^n/n!$ for $y \geq 0$ and 0 for $y \leq 0$ with $n > -1$. Then $u(y,s) = s^{n/2}G_n(\alpha)$ where $G_n(\alpha)$ is a solution of (4.5) and is given by

$$(4.12) \qquad G_n(\alpha) = \frac{1}{n!}\int_{-\alpha}^{\infty} \phi(\epsilon)(\alpha+\epsilon)^n d\epsilon = \frac{1}{n!}\int_0^{\infty} \phi(\alpha-\epsilon)\epsilon^n d\epsilon$$

Symmetric and odd solutions of (4.5) are given by

$$(4.13) \qquad G_{ne}(\alpha) = G_n(\alpha) + G_n(-\alpha)$$

and

$$(4.14) \qquad G_{no}(\alpha) = G_n(\alpha) - G_n(-\alpha)$$

respectively. These correspond to $h(y) = |y|^n/n!$ and $h(y) = \text{sgn}(y)|y|^n/n!$ for all y. Also $G_n'(\alpha) = G_{n-1}(\alpha)$ for $n > 0$. By integration we see that

$$(4.15) \qquad G_0(\alpha) = \Phi(\alpha)$$

and hence

$$(4.16) \qquad G_{0o}(\alpha) = 2[\Phi(\alpha) - 1/2]$$

while

$$(4.17) \qquad G_{0e}(\alpha) = 1$$

Also

$$G_1(\alpha) = \phi(\alpha) + \alpha\Phi(\alpha) \tag{4.18}$$

Using the fact that $G_{n+1}(\alpha) = \int G_n(\alpha)d\alpha$ and $G_n(\alpha) \to 0$ as $\alpha \to -\infty$ it follows that for integer $n > 0$, $G_n(\alpha)$ is of the form (4.1) where $P_n(\alpha)$ is a polynomial of degree $n-1$ and $Q_n(\alpha)$ is a polynomial of degree n. It is easy to see that $P_n(\alpha)$ and $Q_n(\alpha)$ are alternately even and odd polynomials in α of degree $n-1$ and n respectively with leading coefficients $1/n!$. Also for nonnegative integers m

$$G_{2m,e}(\alpha) = Q_{2m}(\alpha) \tag{4.19}$$

$$G_{2m+1,o}(\alpha) = Q_{2m+1}(\alpha) \tag{4.20}$$

$$G_{2m,o}(\alpha) = 2\{P_{2m}(\alpha)\phi(\alpha) + Q_{2m}(\alpha)[\Phi(\alpha) - 1/2]\} \tag{4.21}$$

$$G_{2m+1,e}(\alpha) = 2\{P_{2m+1}(\alpha)\phi(\alpha) + Q_{2m+1}(\alpha)[\Phi(\alpha) - 1/2]\} \tag{4.22}$$

Thus, for positive integer n, $s^{n/2}Q_n(\alpha)$ is a solution of the heat equation which is a polynomial in s and y of degree n in y. For example

$$s^0 Q_0(\alpha) = 1 \ ,$$
$$s^{1/2}Q_1(\alpha) = s^{1/2}\alpha = y$$

and

$$sQ_2(\alpha) = s(\alpha^2+1)/2 = (y^2+s)/2 \ .$$

Another way of looking at $Q_n(\alpha)$ is that

$$Q_n(\alpha) = G_n(\alpha) + (-1)^n G_n(-\alpha) = \frac{1}{n!}E(\alpha+W)^n \tag{4.23}$$

where $\mathcal{L}(W) = N(0,1)$. Thus, for integer $n \geq 0$

$$\begin{aligned} Q_n(\alpha) &= \sum_{j=0}^{[n/2]} \binom{n}{2j} \frac{(2j)!}{2^j j!} \frac{\alpha^{n-2j}}{n!} \\ &= \sum_{j=0}^{[n/2]} \frac{1}{(n-2j)!} \frac{1}{2^j j!} \alpha^{n-2j} \end{aligned} \tag{4.24}$$

In particular for $n = 2m$

$$G_{2m,e}(0) = Q_{2m}(0) = \frac{1}{2^m m!} = \frac{1 \cdot 3 \cdot 5 \cdots (n-1)}{n!} \, . \tag{4.25}$$

Note that for $n = (2m+1)$

$$\begin{aligned} G_{2m+1,e}(0) &= 2\phi(0) P_{2m+1}(0) \\ &= \frac{2\phi(0)}{(2m+1)!} \int_0^\infty e^{-w^2/2} w^{2m+1} dw \, . \end{aligned} \tag{4.26}$$

Let $v = w^2/2$. Then

$$G_{2m+1,e}(0) = \frac{2\phi(0)}{(2m+1)!} \int_0^\infty e^{-v} (2v)^m dv$$

$$G_{2m+1,e}(0) = 2\phi(0) \cdot \frac{2^m m!}{(2m+1)!} = \frac{2\phi(0)}{1 \cdot 3 \cdot 5 \cdots (2m+1)}$$

and

$$P_{2m+1}(0) = \frac{1}{1 \cdot 3 \cdot 5 \cdots (2m+1)} \, . \tag{4.27}$$

It is obvious that $G_{ne}(\alpha) = G_{ne}(0) H_{ne}(\alpha)$. Similarly $G_{no}(\alpha)$ is a multiple of $H_{no}(\alpha)$. That multiple is clearly $G'_{no}(0) = G_{n-1,e}(0)$, and $G_{no}(\alpha) = G_{n-1,e}(0) H_{no}(\alpha)$.

The polynomials $P_n(\alpha)$ can be calculated in a number of ways. Direct integration is one way. Using the equation $G'_n(\alpha) = G_{n-1}(\alpha)$ which implies that

$$P_{n-1}(\alpha)\phi(\alpha) + Q_{n-1}(\alpha)\Phi(\alpha) = \phi(\alpha)[P'_n(\alpha) - \alpha P_n(\alpha) + Q_n(\alpha)] + Q'_n(\alpha)\Phi(\alpha) \tag{4.28}$$

gives an equation for $P_n(\alpha)$ in terms of P_{n-1}, P'_n and Q_n. We can equate coefficients using the fact that $P_n(0)$ or $P'_n(0)$ is known or that the leading coefficient is $1/n!$. Perhaps the most efficient method is the following indirect one which yields $P_n(\alpha)$ as the byproduct of an asymptotic result.

Since

$$\begin{aligned} n! G_n(-\alpha) &= \int_0^\infty v^n \phi(v+\alpha) dv \\ &= \phi(\alpha) \int_0^\infty e^{-v^2/2} e^{-v\alpha} v^n dv \\ &= \frac{\phi(\alpha)}{\alpha^{n+1}} \int_0^\infty e^{-v} e^{-v^2/2\alpha^2} v^n dv \\ &= \frac{\phi(\alpha)}{\alpha^{n+1}} \int_0^\infty e^{-v} \sum_{j=0}^\infty (-1)^j \frac{v^{2j}}{2^j j! \alpha^{2j}} v^n dv \end{aligned} \tag{4.29}$$

we have, as $\alpha \to \infty$

$$(4.30) \qquad G_n(-\alpha) \sim \frac{\phi(\alpha)}{n!} \sum_{j=0}^{\infty} (-1)^j \frac{(2j+n)!}{2^j j!} \alpha^{-(n+1+2j)} .$$

For $n = 0$ this implies the well known asymptotic nonconvergent expansion as $\alpha \to \infty$

$$(4.31) \qquad \frac{\Phi(-\alpha)}{\phi(\alpha)} \sim \frac{1}{\alpha} - \frac{1}{\alpha^3} + \frac{3}{\alpha^5} + \ldots \sim \sum_{j=0} (-1)^j \frac{(2j)!}{2^j j!} \alpha^{-(2j+1)} .$$

Using this expansion, it follows that the nonzero coefficients of $n!P_n(\alpha)$ can be calculated as the first $[(n+1)/2]$ coefficients of the formal expansion of

$$(4.32) \qquad p_n(v) \sim q_n(v) a(v)$$

where

$$(4.33) \qquad a(v) \sim \sum_{j=0}^{\infty} (-1)^j \frac{(2j)!}{2^j j!} v^j$$

and

$$(4.34) \qquad q_n(v) = \sum_{j=0}^{[n/2]} \binom{n}{2j} \frac{(2j)!}{2^j j!} v^j$$

The coefficients of $P_n(\alpha)$ and $Q_n(\alpha)$ are listed in Tables 7a and 7b.

Because $G_0(\alpha) = \Phi(\alpha)$, $G_o'(\alpha) = \phi(\alpha)$ is an even solution of (4.5) for $n = -1$. But the even solution for $n = 0$ is the constant 1 and its derivative does not provide a second independent solution for $n = -1$. However it is easy to see that that is furnished by $J(\alpha)$, which is defined in (4.2), and is odd. Thus successive solutions, alternating even and odd, for negative integer values of n, are derived from successive derivatives of ϕ and J. The derivatives of ϕ can be expressed by

$$(4.35) \qquad \phi^{(m)}(\alpha) = R_m(\alpha) \phi(\alpha)$$

where $R_m(\alpha)$ is a polynomial of degree m with leading coefficient $(-1)^m$, and $\phi^{(m)}$ corresponds to $H_n(\alpha)$ for $n = -(m+1)$. In fact the $R_m(\alpha)$ are the Hermite polynomials except for the factor $(-1)^m$. Substituting in (4.5) we have

$$(4.36) \qquad R_m''(\alpha) - \alpha R_m'(\alpha) = -m R_m(\alpha)$$

and expanding R_m, for even m, as

(4.37) $$R_m(\alpha) = \sum a_{mj}\alpha^{2j}$$

We have

(4.38) $$a_{m,j} = \frac{-[m+2-2j]}{2j(2j-1)} a_{m,j-1}$$

which implies that the $a_{mj}/a_{m0} = (-1)^j c_{mj}$ for even m. Similarly for odd m,

(4.39) $$R_m(\alpha) = \sum a^*_{mj}\alpha^{2j+1}$$

and

(4.40) $$a^*_{mj} = \frac{-[m+1-2j]}{(2j)(2j+1)} a^*_{m,j-1}$$

and $a^*_{mj}/a^*_{m0} = (-1)^j c^*_{mj}$ for odd j also.

For positive integers m, the coefficients of $R_m(\alpha)$ are proportional, with sign changing, to the coefficients of $Q_m(\alpha)$. Since the leading coefficients of $R_m(\alpha)$ alternate between +1 and -1 and those of $m!Q_m(\alpha)$ are 1, it is clear that the coefficients of $m!Q_(\alpha)$ are simply the absolute values of those of the Hermite polynomials and

(4.41) $$R_m(\alpha) = (-1)^m \sum_{j=0}^{[m/2]} (-1)^j \frac{m!}{(m-2j)!} \frac{1}{2^j j!} \alpha^{m-2j} .$$

We turn to $J(\alpha)$. As $\alpha \to 0$

$$J(\alpha) \sim (1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{8} - \ldots) \int_0^\alpha [1 + \frac{x^2}{2} + \frac{x^4}{8} + \ldots]dx$$

(4.42) $$J(\alpha) \sim \alpha - \frac{\alpha^3}{3} + \frac{\alpha^5}{15} - \ldots$$

This expansion coincides with that of $H_{-1,0}(\alpha) = \alpha F(-1, 3/2; -\alpha^2/2)$ corresponding to (4.11). For $\alpha \to \infty$

$$J(\alpha) = \int_0^\alpha e^{(x^2-\alpha^2)/2} dx = \int_0^\alpha e^{[(\alpha-u)^2-\alpha^2]/2} du$$

$$= \int_0^\alpha e^{-\alpha u + u^2/2} du$$

$$= \alpha^{-1} \int_0^{\alpha^2} e^{-v+v^2/2u^2} dv$$

$$= \alpha^{-1} \int_0^{\alpha^2} e^{-v} \left[1 + \frac{v^2}{2\alpha^2} + \frac{v^4}{8\alpha^4} + \ldots\right]$$

$$J(\alpha) \sim \alpha^{-1}\{1 + \alpha^{-2} + 3\alpha^{-4} + 15\alpha^{-6} + \ldots\} \tag{4.43}$$

From (4.2) it is easy to see that

$$J'(\alpha) = 1 - \alpha J(\alpha) \tag{4.44}$$

$$J''(\alpha) = -\alpha + (\alpha^2 - 1)J(\alpha) \tag{4.45}$$

$$J^{(3)}(\alpha) = (\alpha^2 - 2) - (\alpha^3 - 3\alpha)J(\alpha) \tag{4.46}$$

Thus

$$J^{(m)}(\alpha) = S_m(\alpha) + T_m(\alpha)J(\alpha) \tag{4.47}$$

where $S_m(\alpha)$ and $T_m(\alpha)$ are polynomials, the coefficients of which evidently relate to those of $m!P_m(\alpha)$ and $m!Q_m(\alpha)$ except for sign changes. Substituting in (4.5), we have

$$T''_m(\alpha) - \alpha T'_m(\alpha) = -mT_m(\alpha) \tag{4.48}$$

from which it follows that $T_m(\alpha) = R_m(\alpha)$ because the leading coefficients coincide.

By formal differentiation of (4.43) we have, as $\alpha \to \infty$

$$J^{(m)}(\alpha) \sim (-1)^m \alpha^{-m-1} \sum_{j=0}^{\infty} \frac{(2j+m)!}{(2j)!} \frac{(2j)!}{2^j j!} \alpha^{-2j} \tag{4.49}$$

Thus the coefficients of $S_m(\alpha)$ are $(-1)^{m-1}$ times the first $[(m+1)/2]$ of the formal expansion of

$$t_m(v) j_m(v)$$

where

$$j_m(v) = \sum_{j=0}^{\infty} \frac{(2j)!}{2^j j!} v^j \tag{4.50}$$

and

$$t_m(v) = \sum_{j=0}^{[m/2]} (-1)^j \binom{m}{2j} \frac{(2j)!}{2^j j!} v^j \tag{4.51}$$

Relating these expansions to those of (4.32) - (4.34), we see why the coefficients of $S_m(\alpha)$ coincide, except for sign, with those of $m!P_m(\alpha)$. The coefficients of $R_m = T_m$ and S_m are given in Table 8.

Comparing G_{ne}, G_{no}, $J^{(m)}$ and $\phi^{(m)}$ with the confluent hypergeometric functions, we have

$$G_{ne}(\alpha) = G_{ne}(0)F\left(-\frac{n}{2}, \frac{1}{2}; -\frac{\alpha^2}{2}\right)$$

$$G_{no}(\alpha) = G_{n-1,e}(0)\alpha F\left(-\frac{n-1}{2}, \frac{3}{2}; -\frac{\alpha^2}{2}\right)$$

$$J^{(n)}(\alpha) = J^{(n)}(0)F\left(\frac{n+1}{2}, \frac{1}{2}; -\frac{\alpha^2}{2}\right) \qquad \text{for } n \text{ odd}$$

$$J^{(n-1)}(\alpha) = J^{(n)}(0)\alpha F\left(\frac{n+1}{2}, \frac{3}{2}; -\frac{\alpha^2}{2}\right) \qquad ''$$

$$\phi^{(n)}(\alpha) = \phi^{(n+1)}(0)\alpha F\left(\frac{n+2}{2}, \frac{3}{2}; -\frac{\alpha^2}{2}\right) \qquad ''$$

$$\phi^{(n_1)}(\alpha) = \phi^{(n+1)}(0)F\left(\frac{n+2}{2}, \frac{1}{2}; -\frac{\alpha^2}{2}\right) \qquad '' \tag{4.52}$$

where

$$G_{2m,e}(0) = \frac{1}{m!2^m}$$

$$G_{2m-1,e}(0) = \frac{2\phi_0}{1\cdot 3\cdot 5\cdots(2m+1)}$$

$$J^{(2m+1)}(0) = \frac{(-1)^m}{m!2^m}$$

$$\phi^{(2m)}(0) = \frac{(-1)^m\phi_0}{1\cdot 3\cdot 5\cdot\cdots(2m-1)} \tag{4.53}$$

and ϕ_0 is an abbreviation of $\phi(0) = (2\pi)^{-1/2}$.

For our asymptotic expansions for large s in Section 6.3 we will use a variation of Equation 4.3 involving functions which behave like $h(y) = y^{-2}\log y^2$ for large y. Equation (4.3) corresponds to the effect at time s of a distribution of heat given by h. The variation we propose,

$$u(y,s) = s^{-1/2}\int_{-\infty}^{\infty}\left\{[\phi(\frac{w-y}{\sqrt{s}}) + \phi(\frac{w+y}{\sqrt{s}})] - 2\phi(\frac{y}{\sqrt{s})}\right\}h(w)dw$$

$$= \int_{-\infty}^{\infty}\left\{[\phi(\alpha'-\alpha) + \phi(\alpha'+\alpha)] - 2\phi(\alpha)\right\}h(\alpha'\sqrt{s})d\alpha' \tag{4.54}$$

corresponds to a distribution of heat analagous to a source and compensating sink at $y = 0$. This variation permits us to deal with the behavior of h for large s when α' is small. Oddly enough, the behavior of h for small values of y is not important and h can be modified to be zero when $|y| \leq 1$ with little overall effect. It is easy to see that for large s, we obtain

$$(4.55) \qquad u(y,s) \sim h(s^{1/2}\alpha) + \frac{s}{2}h^{(2)}(s^{1/2}\alpha) + \frac{s^2}{8}h^{(4)}(s^{1/2}\alpha) + \frac{s^3}{48}h^{(6)}(s^{1/2}\alpha) + \dots$$

and

$$(4.56) \qquad u_y(y,s) \sim h'(s^{1/2}\alpha) + \frac{s}{2}h^{(3)}(s^{1/2}\alpha) + \frac{s^2}{8}h^{(5)}(s^{1/2}(\alpha) + \dots .$$

Finally, a representation of the solution of the heat equation from Goursat (1942), which was used in Chernoff (1965a) to justify the asymptotic expansions for large s, is reproduced below. We have

$$(4.57) \quad u(y,s) = \int_B ((s-s_1)^{-1/2}\phi\left(\frac{y-y_1}{(s-s_1)^{1/2}}\right)\left[u(y_1,s_1)dy_1 + \frac{1}{2}u_y ds_1 - \frac{u}{2}\left[\frac{y-y_1}{s-s_1}\right]ds_1\right]$$

where the integral is taken along a (y_1, s_1) path B which starts at $s_1 = s$ with a value of y_1 less than y, passes through points $s_1 < s$ and terminates at $s_1 = s$ with a value of y_1 greater than y. The region that B circumscribes is one where u satisfies the heat equation. Equation (4.3) may be regarded as a limiting case.

4.1 Tables.

We present some tables relevant to the solutions of the heat equation. In Table 1 we present the even solution

$$(4.1.1) \qquad H_{ne}(\alpha) = F\left(-\frac{n}{2}, \frac{1}{2}; -\frac{\alpha^2}{2}\right)$$

and in Table 2 the odd solution

$$(4.1.2) \qquad H_{no}(\alpha) = \alpha F\left(-\frac{n-1}{2}, \frac{3}{2}; -\frac{\alpha^2}{2}\right)$$

of

$$(4.1.3) \qquad H_n''(\alpha) + \alpha H'(\alpha) = nH_n(\alpha)$$

as power series expansions in α, with leading coefficient 1. These are related, for positive values of n, to

(4.1.4) $$G_{ne}(\alpha) = G_n(\alpha) + G_n(-\alpha)$$

and

(4.1.5) $$G_{no}(\alpha) = G_n(\alpha) - G_n(-\alpha)$$

These appear alternately in Tables 3 and 4. Table 3 presents

(4.1.6) $$Q_n(\alpha) = G_n(\alpha) + (-1)^n G_n(-\alpha)$$

and Table 4 gives the expansion for

(4.1.7) $$V_n(\alpha) = [G_n(\alpha) + (-1)^{n+1} G_n(-\alpha)]/2\phi_0.$$

For negative values of n, H_{ne} and H_{no} are related to $J^{(-n-1)}(\alpha)$ and $\phi^{(-n-1)}(\alpha)$, for which the expansions appear in Tables 5 and 6.

The relations

(4.1.8) $$G_n(\alpha) = P_n(\alpha)\phi(\alpha) + Q_n(\alpha)\Phi(\alpha)$$

(4.1.9) $$J^{(n)}(\alpha) = R_n(\alpha)J(\alpha) + S_n(\alpha)$$

and

(4.1.10) $$\phi^{(n)}(\alpha) = R_n(\alpha)\phi(\alpha) \ ,$$

where P_n, Q_n, R_n and S_n are polynomials in α, lead to the Tables 7 and 8 for P_n, Q_n, $n!P_n$ $n!Q_n$, R_n and S_n. Note that the coefficients of P_n and Q_n are the absolute values of those for R_n and S_n, and that the $(-1)^n R_n$ are Hermite polynomials. Also $Q_n(\alpha)$ is a solution of (4.1.3) and thus $s^{n/2}Q_n(\alpha)$ satisfies the heat equation. The asymptotic expansions, for $\alpha \to \infty$,

(4.1.11) $$G_n(-\alpha) \sim \frac{\phi(\alpha)}{n!}\sum_{j=0}^{\infty}(-1)^j \frac{(2j+n)!}{2^j j!}\alpha^{-(n+1+2j)}$$

and

(4.1.12) $$J(\alpha) \sim \alpha^{-1}\{1 + \alpha^{-2} + 3\alpha^{-4} + 15\alpha^{-6} + \ldots\}$$

give rise to Table 9 which presents the asymptotic expansions for large α of $G_n(-\alpha)$ and $J^{(n)}(\alpha)$. Equation (4.1.10) is adequate to deal with $\Phi^{(n)}(\alpha)$ for large α, and $G_n(\alpha)$ is described, for large α, by

(4.1.13) $$G_n(\alpha) = Q_n(\alpha) - (-1)^n G_n(-\alpha)$$

Finally, in Table 10 we indicate how H_{ne}, H_{no}, G_{ne}, G_{no}, $J^{(n)}$ and $\phi^{(n)}$ are related.

Table 1. The coefficients a_{nr} of $H_{ne}(\alpha) = \sum_{r=0}^{\infty} a_{nr}\alpha^{2r}$.

$n \backslash r$	0	1	2	3	4
-10	1	-10/2	120/24	-1,680/720	26,880/40,320
-9	1	-9/2	99/24	-1,287/720	19,305/40,320
-8	1	-8/2	80/24	-960/720	13,440/40,320
-7	1	-7/1	63/24	-693/720	9,009/40,320
-6	1	-6/2	48/24	-480/720	5,760/40,320
-5	1	-5/2	35/24	-315/720	3,465/40,320
-4	1	-4/2	24/24	-192/720	1,920/40,320
-3	1	-3/2	15/24	-105/720	945/40,320
-2	1	-2/2	8/24	-48/720	384/40,320
-1	1	-1/2	3/24	-15/720	105/430
0	1	0	0	0	0
1	1	1/2	-1/24	3/720	-15/40,320
2	1	2/2	0	0	0
3	1	3/2	3/24	-3/720	9/40,320
4	1	4/2	8/24	0	0
5	1	5/2	15/24	15/720	-15/40,320
6	1	6/2	24/24	48/720	0
7	1	7/2	35/24	105/720	105/40,320
8	1	8/2	48/24	192/720	384/40,320
9	1	9/2	63/24	315/720	945/40,320
10	1	10/2	80/24	480/720	1,920/40,320

Table 2. The coefficients a_{nr} of $H_{no}(\alpha) = \sum_{r=0}^{\infty} a_{nr} \alpha^{1+2r}$.

$n \backslash r$	0	1	2	3	4
-10	1	-11/6	143/120	-2,145/5,040	36,465/362,880
-9	1	-10/6	120/120	-1,680/5,040	26,880/362,880
-8	1	-9/6	99/120	-1,287/5,040	19,305/362,880
-7	1	-8/6	80/120	-960/5,040	13,440/362,880
-6	1	-7/6	63/120	-693/5,040	9,009/362,880
-5	1	-6/6	48/120	-480/5,040	5,760/362,880
-4	1	-5/6	35/120	-315/5,040	3,465/362,880
-3	1	-4/6	24/120	-192/5,040	1,920/362,880
-2	1	-3/6	15/120	-105/5,040	945/362,880
-1	1	-2/6	8/120	-48/5,040	384/362,880
0	1	-1/6	3/120	-15/5,040	105/362,880
1	1	0	0	0	0
2	1	1/6	-1/120	3/5,404	-15/362,880
3	1	2/6	0	0	0
4	1	3/6	3/120	-3/5,040	9/362,880
5	1	4/6	8/120	0	0
6	1	5/6	15/120	15/5,040	-15/362,880
7	1	6/6	24/120	48/5,040	0
8	1	7/6	35/120	105/5,040	105/362,880
9	1	8/6	48/120	192/5,040	384/362,880
10	1	9/6	63/120	315/5,040	945/362,880

Table 3 $G_{ne}(\alpha) = Q_n(\alpha) = G_n(\alpha) + (-1)^n G_n(-\alpha) = \sum a_{nr}\alpha^r = \frac{1}{n!}\sum_{r=0}^{n} b_{nr}\alpha^r$.

a_{nr}

n\ r	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		1									
2	1/2		1/2								
3		1/2		1/6							
4	1/8		1/4		1/24						
5		1/8		1/2		1/120					
6	1/48		1/16		1/48		1/720				
7		1/48		1/48		1/240		1/5,040			
8	1/384		1/96		1/192		1/1440		1/40,320		
9		1/384		1/288		1/960		1/10,080		1/362,880	
10	1/3,840		1/768		1/1,152		1/5,760		1/80,640		1/3,628,800

$b_{nr} = n!a_{nr}$

n\r	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		1									
2	1		1								
3		3		1							
4	3		6		1						
5		15		10		1					
6	15		45		15		1				
7		105		105		21		1			
8	105		420		210		28		1		
9		945		1,260		378		36		1	
10	945		4,725		3,150		630		45		1

Table 4: $T_n(a) = G_{no}(\alpha)2\phi_0 = [G_n(\alpha) + (-1)^{n+1}G_n(-\alpha)]/2\phi_0 = \sum_{r=0}^{\infty} a_{nr}\alpha^r$

	a_{nr}									
n\r	0	1	2	3	4	5	6	7	8	9
0		1		-1/6		1/40		-1/336		1/3,456
1	1		1/2		-1/24		1/240		-1/2,688	
2		1		1/6		-1/120		1/1,680		-1/24,192
3	1/3		1/2		1/24		-1/720		1/13,440	
4		1/3		1/6		1/120		-1/5,040		1/120,960
5	1/15		1/6		1/24		1/720		1/40,320	
6		1/15		1/18		1/120		1/5,040		-1/362,880
7	1/105		1/30		1/72		1/720		1/40,320	
8		1/105		1/90		1/360		1/5,040		1,362,880
9	1/945		1/210		1/360		1,2,160		1/40,320	
10		1/945		1/630		1/1,800		1/15,120		1/362,880
11	1/10,395		1/1,890		1/2,520		1/10,800		1/120,960	

Table 5: $J^{(n)}(\alpha) = \sum_{j=0}^{\infty} a_{nr} \alpha^{r}$

					a_{nr}					
n\ r	0	1	2	3	4	5	6	7	8	9
0		1		-1/3		1/15		-1/105		1/945
1	1		-1		1/3		-1/15		1/105	
2		-2		4/3		-2/5		8/105		-2/189
3	-2		4		-2		8/15		-2/21	
4		8		-8		16/5		-16/21		8/63
5	8		-24		16		-16/3		8/7	
6		-48		64		-32		32		-16/9

Table 6: $\phi^{(n)}(\alpha) = \phi_0 \sum_{r=0}^{\infty} a_{nr} \alpha^r$

a_{nr}

n\r	0	1	2	3	4	5	6	7	8	9
0	1		-1/2		1/8		-1/48		1/384	
1		-1		1/2		-1/8		1/48		-1/384
2	-1		3/2		-5/8		7/48		-3/128	
3		3		-5/2		7/8		-3/16		11/384
4	3		-15/2		35/8		-21/16		33/128	
5		-15		35/2		-63/8		33/16		-143/384
6	-15		105/2		-315/8		231/16		-429/128	

Table 7: $G_n(\alpha) = P_n(\alpha)\phi(\alpha) + Q_n(\alpha)\Phi(\alpha)$

Table 7a: $Q_n(\alpha) = \sum_{r=0}^{n} a_{nr}\alpha^r = \frac{1}{n!}\sum_{r=0}^{m} b_{nr}\alpha^r$

a_{nr}

n\r	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		1									
2	1/2		1/2								
3		1/2		1/6							
4	1/8		1/4		1/24						
5		1/8		1/12		1/120					
6	1/48		1/16		1/48		1/720				
7		1/48		1/48		1/240		1/5,040			
8	1/384		1/96		1/192		1/1,440		1/40,320		
9		1/384		1/288		1/960		1/10,080		1/362,880	
10	1/3,840		1/768		1/1,152		1/5,760		1/80,640		1/3,628,800

b_{nr}

n\r	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		1									
2	1		1								
3		3		1							
4	3		6		1						
5		15		10		1					
6	15		45		15		1				
7		105		105		21		1			
8	105		420		210		28		1		
9		945		1,260		378		36		1	
10	945		4,725		3,150		630		45		1

Table 7b: $P_n(\alpha) = \sum_{r=0}^{n-1} a^*_{nr}\alpha^r = \frac{1}{n!}\sum_{r=0}^{n-1} b^*_{nr}\alpha^r$

a^*_{nr}

n\r	0	1	2	3	4	5	6	7	8	9
1	1									
2		1/2								
3	1/3		1/6							
4		5/24		1/24						
5	1/15		3/40		1/120					
6		11/240		14/720		1/720				
7	1/105		87/5,040		1/252		1/5,040			
8		31/4,480		37/8,064		3/4,480		1/40,320		
9	1/945		65/24,192		23/24,192		1/10,368		1/362,880	
10		193/241,920		11/15,120		7/43,200		11/907,200		1/3,628,800

b^*_{nr}

n\r	0	1	2	3	4	5	6	7	8	9
1	1									
2		1								
3	2		1							
4		5		1						
5	8		9		1					
6		33		14		1				
7	48		87		20		1			
8		279		185		27		1		
9	384		975		345		35		1	
10		2,895		2,640		588		44		1

Table 8.

$$\phi^{(n)}(\alpha) = R_n(\alpha)\phi(\alpha)$$

$$J^{(n)}(\alpha) = R_n(\alpha)J(\alpha) + S_n(\alpha)$$

$$R_n(\alpha) = \sum_0^n a_{nr}\alpha^2, \; S_n(\alpha) = \sum_0^{n-1} b_{nr}\alpha^r$$

a_{nr}

n\r	0	1	2	3	4	5	6	7	8	9	10
0	1										
1		-1									
2	-1		1								
3		3		-1							
4	3		-6		1						
5		-15		10		-1					
6	-15		45		-15		1				
7		105		-105		21		1			
8	105		-420		210		-28		1		
9		-945		1,260		-378		36		1	
10	-945		4,725		-3,150		1,640		-45		1

b_{nr}

n\r	0	1	2	3	4	5	6	7	8	9	10
1	1										
2		-1									
3	-2		1								
4		5		-1							
5	8		-9		1						
6		-33		14		-1					
7	-48		87		-20		1				
8		279		-185		27		-1			
9	384		-976		345		-35		1		
10		-2,895		2,640		-588		44		1	

Table 9. Asymptotic Expansions for large α for $J^{(n)}(\alpha)$ and $G_n(-\alpha)$

$$J^{(n)}(\alpha) \sim \alpha^{-(n+1)} \Sigma a_{nr} \alpha^{-2r}$$

$$G_n(\alpha) \sim \alpha^{-(n+1)} \phi(\alpha) \Sigma b_{nr} \alpha^{-2r}$$

a_{nr}

n\r	0	1	2	3
0	1	1	3	15
1	-1	-3	-15	-105
2	2	12	90	840
3	-6	-60	-630	-7,560
4	24	360	5,040	75,600
5	-120	-2,520	-45,360	-831,600
6	720	20,160		
7	-5,040	-181,400		
8	40,320			
9	-362,880			

b_{nr}

n\r	0	1	2	3
0	1	-1	3	-15
1	1	-3	15	-105
2	1	-6	45	-420
3	1	-10	105	-1,260
4	1	-15	210	-3,150
5	1	-21	378	-6,930
6	1	-28	630	-13,860

Table 10. Relations among H, J, ϕ, G and T

$$
\begin{array}{ll}
H_{-2m,e} = J^{(2m-1)}/a_{2(m-1)} & H_{2m,e} = |a_{2m}|G_{2m,e} = |a_{2m}|Q_{2m} \\
H_{-2m-1,0} = J^{(2m)}/a_{2m} & H_{2m+1,0} = |a_{2m}|G_{2mH,0} = |a_{2m}|Q_{2m+1} \\
H_{-2m-1,e} = \phi^{(2m)}/b_{2m}\phi_0 & H_{2m-1,e} = |b_{2m}|G_{2m-1,e}2\phi_0 = |b_{2m}|T_{2m-1} \\
H_{-2m,0} = \phi^{(2m-1)}/b_{2m}\phi_0 & H_{2m,0} = |b_{2m}|G_{2m-1,e}2\phi_0 = |b_{2m}|T_{2m}
\end{array}
$$

n	0	2	4	6	8	10
a_n	1	-2	8	-48	384	-3,840
b_n	1	-1	3	-15	105	-945

$$a_{2m} = (-1)^m 2^m m! \quad b_{2m} = (-1)^m \frac{(2m)!}{2^m m!} = (-1)^m (1 \cdot 3 \cdots (2m-1))$$

5. Bounds

In this section we derive some qualitative properties of the optimal sequential rules for Problems 1 and 2. Most of these establish the bounds described in Section 3. First we have

Proposition 1. The optimal continuation set $\tilde{C}$ decreases with c_1 in Problem 1, and with c_2 in Problem 2.

Proof: Let d_{12}, ρ and $\tilde{C}$ correspond to c_1 and d_{12}^*, ρ^*, and $\tilde{C}^*$ to $c_1^* > c$ and let $(y, s) \in \tilde{C}^*$. Then there is a stopping rule $S < s$ for which

$$\begin{aligned} d_{12}^*(y,s) &> E\{d_{12}^*[Y(S), S] | Y(s) = y\} \\ &= E\{d_{12}[Y(S), S] + (c_1^* - c_1)S^{-1} | Y(s) = y\} \\ &\geq \rho(y,s) + (c_1^* - c_1)s^{-1} \end{aligned}$$

and

$$d_{12}(y,s) = d_{12}^*(y,s) - (c_1^* - c_1)s^{-1} > \rho(y,s)$$

Hence, $(y,s) \in \tilde{C}$ and $\tilde{C} \supset \tilde{C}^*$. The same proof applies for Problem 2.

∎

We state two related propositions and a more general lemma, on which the proofs of the propositions are based.

Proposition 2. In Problem 1 with stopping cost d_{12}, the optimal continuation set $\tilde{C}_1$ contains $\{(y,s):\ y^2 + s = 1,\ s > 0,\ y \neq 0\}$.

Proposition 2*. In Problem 2 with stopping cost d_{22}, the optimal continuation set $\tilde{C}_2$ contains $\{(y,s): s^{1/2}G_{1e}(ys^{-1/2}) = 1,\ s > 0,\ y \neq 0\}$.

Suppose that $y_0(s)$ is a differentiable curve along which $d(y,s)$ is continuous and differentiable, but along which the right hand and left hand derivatives with respect to y, d_y^+ and d_y^- are defined finite but unequal.

Lemma 1 **For** $\mathcal{L}(W) = N(0,1)$, $E\{d[y_0(s) + W\sqrt{\delta},\ s - \delta] = d[y_0(s), s] + \phi_0(d_y^+ - d_y^-)\sqrt{\delta} + o(\sqrt{\delta})$.

Proof:

$$\begin{aligned} d[y_0(s) + W\sqrt{\delta}, s-\delta] &= d[y_0(s-\delta) + W\sqrt{\delta} + y_0(s) - y_0(s-\delta), s-\delta] \\ &= d[y_0(s), s] + [W\sqrt{\delta}]d_y^+ + O(\delta) \qquad \text{for } W > O(\sqrt{\delta}) \\ &= d[y_0(s), s] + [W\sqrt{\delta}]d_y^- + O(\delta) \qquad \text{for } W < O(\sqrt{\delta}) \end{aligned}$$

Hence

$$\begin{aligned} E\{d[y_0(s) + W\sqrt{\delta}, s-\delta]\} &= d[y_0(s), s] + \sqrt{\delta}\{d_y^+ \int_0^\infty w\phi(w)dw + d_y^- \int_{-\infty}^0 w\phi(w)dw\} + O(\delta) \\ &= d[y_0(s), s] + \phi_0\sqrt{\delta}[d_y^+ - d_y^-] + O(\delta) \end{aligned}$$

∎

Note: Implicit in the proof above was the assumption, not clearly stated here, that d does not grow too fast as y goes to $\pm\infty$.

Propositions 2 and 2* are corollaries of Lemma 1. For Proposition 2, $d_{12y}^+ = 0$, and $d_{12y}^- = 2y$ for $y^2 + s = 1$ and $y > 0$. While $d_{12y}^- = 0$ and $d_{12y}^+ = 2y$ for $y^2 + s = 1$ and $y < 0$. In both cases $d_{12y}^+ - d_{12y}^- < 0$ and the procedure of letting $S = s - \delta$ leads to a reduction of risk over stopping at (y, s).

For Proposition 2* we have for $s^{1/2}G_{1e}(ys^{-1/2}) = 1$ and $y > 0$, $d_{22y}^+ = 0$ and $d_{22y}^- = 2\Phi(ys^{-1/2}) - 1 > 0$ For $y < 0$, $d_{22y}^- = 0$ and $d_{22y}^+ = 2\Phi(ys^{-1/2}) - 1 < 0$. Once more we have $d_{22y}^+ - d_{22y}^- < 0$ in each case and the result follows.

∎

Propositions 2 and 2* fail to determine whether the points $(0, \tilde{s}_1) = (0,1)$ and $(0, \tilde{s}_2) = (0, \pi/2)$, where $\tilde{s}_2^{1/2}G_{1e}(0) = 1$, are, or are not, in the continuation sets. This question is partially addressed by Propositions 3 and 3*.

Proposition 3. **If** $c_1 < (2/\pi e)^{1/2} = 0.484$, $(0,1) \in \tilde{\mathcal{C}}$ **in Problem 1.**

Proposition 3* **If** $c_2 < (\pi/8e)^{1/2} = 0.380$, $(0, \pi/2) \in \tilde{\mathcal{C}}$ **in Problem 2.**

Proof: With $\mathcal{L}(W) = N(0,1)$, we have

$$E\{d_{12}(W\sqrt{\delta}, 1-\delta)\} = \frac{c_1}{1-\delta} + \int_{-\infty}^{\infty} \min(1, w^2\delta + 1 - \delta)\phi(w)dw$$
$$= c_1 + 1 + \delta\{c_1 + \int_{-1}^{1} (w^2 - 1)\phi(w)dw\} + o(\delta)$$
$$= d(0,1) + \delta\{c_1 - 2\phi(1)\} + o(\delta)$$

Hence $(0,1) \in \tilde{C}$ if $c_1 < 2\phi(1)$.

$$d_{22}(W\sqrt{\delta}, \frac{\pi}{2} - \delta) = \frac{c_2}{\pi/2 - \delta} + \min[1, (\frac{\pi}{2} - \delta)^{1/2} G_{1e}(W\delta^{1/2}(\frac{\pi}{2} - \delta)^{-1/2})]$$

Since $G_{1e}(\alpha) = 2\phi_0(1 + \alpha^2/2) + o(\alpha^2)$ for $\alpha \to 0$

$$E\left\{d_{22}(W\sqrt{\delta}, \frac{\pi}{2} - \delta)\right\} = d(0,\pi/2) + \delta E\left\{\frac{4}{\pi^2}c_2 + \min(0, \frac{W^2-1}{\pi})\right\} + o(\delta)$$
$$= d(0,\pi/2) + \delta\left[\frac{4}{\pi^2}c_2 + \int_{-1}^{1}\left(\frac{w^2-1}{\pi}\right)\phi(w)dw\right] + o(\delta)$$

Thus $(0,\pi/2) \in C$ if

$$c_2 < \frac{\pi^2}{4}\frac{1}{\pi}\int_{-1}^{1}(w^2-1)\phi(w)dw = \frac{\pi}{2}\phi(1) = \sqrt{\pi/8e}$$

From a slight perturbation of Propositions 3 and 3* it follows that $\tilde{c}_{1i} \geq (2/\pi e)^{1/2}$ and $\tilde{c}_{2i} \geq (\pi/8e)^{1/2}$. Upper bounds on $\tilde{c}_{1i}$ and $\tilde{c}_{2i}$ follow from Propositions 4 and 4*.

Proposition 4. In Problem 1 all points $(0,s)$ with $0 \leq s \leq c_1^{1/2}$ are in $\tilde{S}$ if $0 < c_1 \leq 1$. If $c_1 > 1$, $(0,s) \in \tilde{S}$ for all $s > 0$, and $\tilde{c}_{1i} \leq 1$.

Proposition 4*. In Problem 2 all points $(0,s)$ with $0 \leq s \leq (c_2/\phi_0)^{2/3}$ are in $\tilde{S}$ if $0 < c_2 \leq \pi/4$. If $c_2 > \pi/4$, $(0,s) \in \tilde{S}$ for all $s > 0$, and hence $\tilde{c}_{2i} \leq \pi/4$.

Proof: For Problem 1, let $0 < s_1 < s \leq c_1^{1/2} \leq 1$. Then

$$d_{12}(0,s) = c_1 s^{-1} + s$$

but for all y

$$d_{12}(y, s_1) \geq c_1 s_1^{-1} + s_1 > d_{12}(0,s)$$

since $c_1 s^{-1} + s$ is decreasing in s for $0 < s \le c_1^{1/2}$. Hence it is disadvantageous to continue from $(0, s)$.

If $c_1 > 1$ and $0 < s_1 < s$, then for all y

$$d_{12}(y, s_1) \ge d_{12}(0, s_1) = c_1 s_1^{-1} + \min(1, s_1)$$

which is a decreasing function of s_1. Hence $d_{12}(y, s_1) \ge d_{12}(0, s)$ and $(0, s) \in \tilde{S}$. It follows immediately that $\tilde{c}_{1i} \le 1$.

For Problem 2, let $0 < s_2 < s \le (2\pi c_2^2)^{1/3} \le \pi/2$. Then

$$d_{22}(0, s) = c_2 s^{-1} + \sqrt{2/\pi}\, s^{1/2}$$

since $G_{1e}(\alpha)$ assumes its minimum value of $2\phi_0 = \sqrt{2/\pi}$ at $\alpha = 0$ and $2\phi_0 s^{1/2} \le 1$. For all y

$$d_{22}(y, s_2) \ge c_2 s_2^{-1} + \sqrt{2/\pi}\, s_2^{1/2} \ge d_{22}(0, s)$$

since $c_2 s^{-1} + \sqrt{2/\pi}\, s^{1/2}$ is decreasing in s for $s \le (2\pi c_2^2)^{1/3}$. Once again it is disadvantageous to continue from $(0, s)$.

If $c_2 > \pi/4$ and $0 < s_2 < s$, then for all y

$$d_{22}(y, s_2) \ge d_{22}(0, s_2) = c_2 s_2^{-1} + \min(1, \sqrt{2/\pi}\, s_2^{1/2})$$

which is a decreasing function of s_2. Thus $d_{22}(y, s_2) \ge d_{22}(0, s)$ and $(0, s) \in \tilde{S}$, and consequently $\tilde{c}_{2i} \le \pi/4$.

∎

Some explicit bounds for $\tilde{s}_{1o}$ and $\tilde{s}_{2o}$ are given in Propositions 5 and 5*.

Proposition 5. For $1/4 < c_1 \le 1$, $(y, s) \in \tilde{S}$ if $s \ge c_1/(2c_1^{1/2} - 1)$.

Proposition 5*. For $2\pi/27 < c_2 \le \pi/4$, $(y, s) \in \tilde{S}$ if

$$s \ge c_2/(3 c_2^{1/3} \phi_0^{2/3} - 1)$$

Proof: Let S be an arbitrary stopping procedure (starting from (y,s)) for Problem 1. Let $E\{X;A\}$ represent the contribution of the event A to EX, i.e., $\int_A X(w)dP(w)$. Then

$$\begin{aligned} B_1 = E\{d_{12}[Y(S),S]\} &= E\{d_{12}[Y(S),S]; S \geq 1\} + E\{d_{12}[Y(S),S]; S < 1\} \\ &\geq (c_1 s^{-1}+1)P\{S \geq 1\} + 2c_1^{1/2}P\{S<1\} \end{aligned}$$

since for $s \geq S \geq 1$, $d_{12}(y,S) = c_1 S^{-1}+1$, and for $0 < S \leq 1$, $d_{12}(y,S) \geq c_1 S^{-1} + S \geq 2c_1^{1/2}$.

As c_1 ranges from $1/4$ to 1, $c_1/(2c_1^{1/2}-1)$ decreases monotonically from ∞ to 1 and hence $s \geq 1$, and $d_{12}(y,s) = c_1 s^{-1}+1$, and

$$B_1 - d_{12}(y,s) \geq (2c_1^{1/2} - c_1 s^{-1} - 1)P\{S<1\} \geq 0,$$

and hence $(y,s) \in \mathcal{S}$.

Let S be an arbitrary stopping procedure for Problem 2. Then

$$\begin{aligned} B_2 = E\{d_{22}[Y(S),S]\} &= E\{d_{22}[Y(S),S]; S \geq \pi/2\} + E\{d_{22}[Y(S),S]; S < \pi/2\} \\ &\geq (c_2 s^{-1}+1)P\{S \geq \pi/2\} + (27c_2/2\pi)^{1/3}P\{S<\pi/2\} \end{aligned}$$

since for $s \geq S \geq \pi/2$, $d_{22}(y,S) \geq c_2 S^{-1}+1$ and for $0 < S \leq \pi/2$, $d_{22}(y,S) \geq c_2 S^{-1} + S^{1/2}G_{1e}(0) = c_2 S^{-1} + 2\phi_0 S^{1/2} \geq (27c_2/2\pi)^{1/3}$.

As c_2 ranges from $2\pi/27$ to $\pi/4$, $c_2/(3c_2^{1/3}\phi_0^{2/3}-1)$ decreases monotonically from ∞ to $\pi/2$ and hence $s \geq \pi/2$ and $d_{22}(y,s) = c_2 s^{-1}+1$, and

$$B_2 - d_{22}(y,s) \geq [(27c_2/2\pi)^{1/3} - c_2 s^{-1} - 1]P\{S<\pi/2\} \geq 0$$

and hence $(y,s) \in \tilde{\mathcal{S}}$.

∎

As a consequence of Propositions 5 and 5*, it follows that $\tilde{s}_{1o} \leq c_1/(2c_1^{1/2}-1)$ and $\tilde{s}_{2o} \leq c_2/(3c_2^{1/3}\phi_0^{2/3}-1)$, $\tilde{c}_{1i} \leq 1$, $\tilde{c}_{2i} \leq \pi/4$, $\tilde{c}_{1o} \leq 1/4$, and $\tilde{c}_{2o} \leq 2\pi/27$.

In Propositions 6 and 6* we derive inner bounds on $\tilde{\mathcal{C}}$ for Problems 1 and 2 for c_1 and c_2 sufficiently small. These incidentally imply that

$$\tilde{c}_{1o} \geq c_{13} = \left\{4 \max_{0 \leq \alpha \leq \alpha_0} (\alpha^2+1)\left[1 - \frac{J'(\alpha)}{J'(\alpha_0)}\right]\right\}^{-1} = 0.0554$$

where

$$J'(\alpha_0) = \inf_{0 \le \alpha < \infty} J'(\alpha) = -0.2847$$

and

$$\alpha_0 = 2.124.$$

Also

$$\tilde{c}_{2o} \ge c_{23} = \frac{4}{27}\left\{\max_{0 \le \alpha \le \alpha_0} G_{1e}^2(\alpha)[1 - J'(\alpha)/J'(\alpha_0)]\right\}^{-1} = 0.0498 .$$

Proposition 6. **For** $0 < c_1 < c_{13}$ **in Problem 1, the optimal continuation set contains** A_1 **which consists of the set of all** (y, s) **for which** $|\alpha| < \alpha_0$ **and**

$$s > \frac{1 + \{1 - 4c_{13}(1-\alpha^2)^{1/2}[1 - J'(\alpha)/J'(\alpha_0)]\}^{1/2}}{2(1+\alpha^2)} .$$

Proposition 6*. **For** $0 < c_2 < c_{23}$ **in Problem 2, the optimal continuation set contains** A_2 **which consists of all** (y, s) **for which** $|\alpha| < \alpha_0$ **and** $s > s_+(\alpha)$ **where** $s_+(\alpha)$ **is the larger of the two positive roots of**

$$s^{3/2}G_{1e}(\alpha) - s + c_{23}[1 - J'(\alpha)/J'(\alpha_0)] = 0 .$$

Proof. Let $u(y,s) = 1 + c_1 s^{-1} J'(\alpha)/J'(\alpha_0)$, the second term of which imitates $c_1 s^{-1}$ when $s \to 0$ and $\alpha = \alpha_0$. Our proof involves showing that $u < d_{12}$ on a set A, and $u = d_{12}$ on the boundary of A. Thus u is the Bayes Risk corresponding to the procedure defined by $C = A$. Since $\rho(y,s) \le u(y,s)$ everywhere, then $\rho < d_{12}$ whenever $u < d_{12}$ and hence the optimal continuation set $\tilde{C}_1 \supset A$. Now $\tilde{C}_1$ is monotone decreasing in c_1. But our set A is monotone increasing in c_1. Hence $\tilde{C}_1$ must contain $A_1 = A(c_{13})$.

Since all points $(0, s)$ for which $s > 1$ are included in C_1, it follows that $\tilde{c}_{1o} \ge c_{13}$. Figure 2 helps to illustrate the details of the proof, which follows.

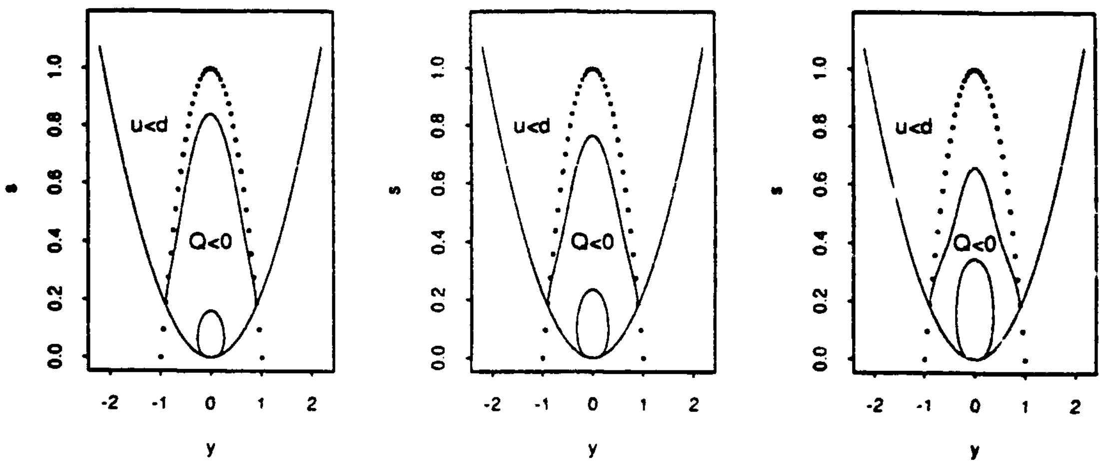
The two curves $y^2 = \alpha_0^2 s$ and $y^2 + s = 1$ intersect at $(\pm y_0, s_0)$ where $s_0 = (\alpha_0^2 + 1)^{-1} < 1$ and $y_0 = \alpha_0 s_0^{1/2}$. The set $B_1 = \{(y,s) : y^2 + s \ge 1,\ y^2 < \alpha_0^2 s\}$ will be part of A. The two points $(\pm y_0, s_0)$ will be connected by a curve segment $s_+(y)$ which together with $s_-(y)$ are the roots of

$$Q(s) = (\alpha^2 + 1)s^2 - s + c_1[1 - J'(\alpha)/J'(\alpha_0)] = 0.$$

These roots are real when the discriminant

$$D = 1 - [illegible]\,\alpha^2 + 1)c_1[1 - J'(\alpha)/J'(\alpha_0)] > 0$$

for $|\alpha| \leq \alpha_0$. That is the case because $0 < c_1 < c_{13}$. Moreover $0 < D \leq 1$ and the roots s_+ and s_- are given by $(1 \pm D^{1/2})/2(1+\alpha^2)$ satisfying $0 \leq s_- < s_+ \leq (\alpha^2+1)^{-1} \leq 1$. Let $A = B_1 \cup B_2$ where $B_2 = \{(y,s) : y^2 + s < 1, y^2 < \alpha_0^2 s, s > s^+\}$.



$c_1 = 0.03$ small $\quad$ $c_1 = 0.04$ medium $\quad$ $c_1 = 0.05$ large

Figure 2. Regions bounded by $|\alpha_0| = \alpha$, $u = d_{12}$, and $Q = 0$ for Problem 1 for 3 values of c_1.

Now we show that $d_{12} - u > 0$ on A with equality on the boundary. On B_1

$$d_{12} - u = c_1 s^{-1}[1 - J'(\alpha)/J'(\alpha_0)] > 0$$

and on B_2, where $s > s_+$

$$d_{12} - u = s^{-1}Q(s) > 0.$$

Equality on the boundary of A is apparent. Thus, $\rho(y,s) \leq u(y,s) < d(y,s)$ on A, $\tilde{C}_1 \supset A$. As c_1 increases subject to $0 < c_1 < c_{13}$, D and s_+ decrease, enlarging B_2 and A which approaches $A_1 = A(c_{13})$ but $\tilde{C}_1$ decreases. Hence $\tilde{C}_1 \supset A_1 = A(c_{13}) \supset A(c_1)$. Proposition 6 follows.

Our proof has incidentally shown that $\tilde{C}_1$ includes all points $(0, s)$ for which $s > 1$ if $c_1 < c_{13}$ and hence $\tilde{c}_{1o} \geq c_{13}$.

The proof of Proposition 6* proceeds along similar lines. We use the same solution u of the heat equation with c_1 replaced by c_2. The two curves $y^2 = \alpha_0^2 s$ and $s^{1/2}G_{1e}(ys^{-1/2}) = 1$ intersect at $(\pm y_0^*, s_0^*)$ where $0 < s_0^* = [G_{1e}(\alpha_0^*)]^{-2} < \pi/2$ and $y_0^* = \alpha_0 s_0^{*1/2}$. For the time being, let us assume that these two points are connected by a curve s_+^* on which

$$Q^*(s, \alpha) = s^{3/2}G_{1e}(\alpha) - s + c_2[1 - J'(\alpha)/J'(\alpha_0)] = 0$$

where $s_+^{*1/2}G_{1e}(ys_+^{*-1/2}) < 1$, and $Q^*(s) > 0$ if $s > s_+^*$, $s^{1/2}G_{1e}(\alpha) < 1$ and $|\alpha| < \alpha_0$. Then we let $A^* = B_1^* \cup B_2^*$ where $B_1^* = \{(y, s) : s^{1/2}G_{1e}(ys^{-1/2}) > 1,\ y^2 \leq \alpha_0^2 s\}$ and $B_2^* = \{(y, s) : s^{1/2}G_{1e}(ys^{-1/2}) < 1,\ y^2 < \alpha_0^2 s,\ s > s_+^*\}$. On B_1^*, $d_{22} - u = c_2 s^{-1}[1 - J'(\alpha)/J'(\alpha_0)] > 0$ and on $B_2^*, d_{22} - u = s^{-1}Q^*(s, \alpha) > 0$. The rest of the proof follows as in that of Proposition 6, provided s_+^* is nonincreasing in c_2. It remains only to demonstrate the existence of the desired s_+^*.

For each fixed α in $(-\alpha_0, \alpha_0)$, $Q^*(s_1^*, \alpha) > 0$ at $s_1^* = 1/G_{1e}^2(\alpha)$. Also

$$\frac{\partial Q^*}{\partial s^*} = \frac{3}{2}s^{1/2}G_{1e}(\alpha) - 1 > 0$$

for $s > s_2^* = 4/9G_{1e}^2(\alpha)$. But

$$Q^*(s_2^*, \alpha) = \frac{-4}{27G_{1e}^2(\alpha)} + c_2\left[1 - \frac{J'(\alpha)}{J'(\alpha_0)}\right] < 0$$

for $c_2 < c_{23}$ by definition of c_{23}. Thus there is a value $s_+^*(\alpha) = \sup\{s : Q^*(s, \alpha) = 0,\ s_2^* < s < s_1^*\}$. Since $\partial Q^*/\partial s > 0$ and $\partial Q^*/\partial \alpha = 2s^{3/2}[\Phi(\alpha) - 1/2] - c_2 J''(\alpha)/J'(\alpha_0)$, for $s > s_2^*$ and $|\alpha| < \alpha_0$, s_+^* defines a curve and one for which s_+^* is decreasing in c_2.

6. Asymptotic Expansions

This section has 4 subsections devoted to asymptotic expansions of the optimal boundaries and risks for Problems 1 and 2 when s is small and when s is large. Most of our derivations will be formal, since the argument required for rigor are complex and have been outlined elsewhere for similar problems, e.g., in Chernoff (1965).

6.1 Problem 1, s small.

The boundary of the optimal stopping set is near $y = \pm 1$. To simplify the calculations we will consider solutions of the heat equation of the form $u(y-1,s)+u(-y-1,s)$ where $u(-y-1,s)$ will have a negligible effect near $y=1$. In effect, for s close to 0 we have transformed the problem to that of dealing with $d_{13}(z,s)$ near $z=0$ where $z=y-1$ and $d_{13}(z,s) = d_{12}(y,s)$, i.e.,

$$\begin{aligned} d_{13}(z,s) &= c_1 s^{-1} + \min(1, z^2+2z+s+1) \\ &= c_1 s^{-1} + 1 + \min[0, s(\beta^2+1) + 2s^{1/2}\beta] \end{aligned} \tag{6.1.1}$$

and $\beta = zs^{-1/2}$. For further convenience we subtract several solutions of the heat equation from d_{13}. First we subtract $c_1 s^{-1} J'(\beta)$ to reduce considerably the unpleasant effect of the s^{-1} singularity when s and β are small. To make the remaining term more symmetric in z, we subtract $1 + s^{1/2}\beta + s(\beta^2+1)/2$ and we have

$$d_{13}(z,s) = c_1 s^{-1} J'(\beta) + 1 + s^{1/2}\beta + s(\beta^2+1)/2 + c_1 d_{14}(z,s) \tag{6.1.2}$$

and

$$d_{14}(z,s) = s^{-1}\left[1 - J'(\beta)\right] - c_1^{-1}|s^{1/2}\beta + s(\beta^2+1)/2| \tag{6.1.3}$$

We will seek approximations to inner and outer boundaries β_i and β_o, that we assume are close to $y^2 + s = 1$ in the original variables, and a solution of the heat equation u so that u and $\partial u/\partial\beta$ match d_{14} and $d_{14\beta} = \partial d_{14}/\partial\beta$ on both boundaries. With an expansion for the boundaries of the form

$$\beta = \frac{-s^{1/2}}{2}\{1 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots\} \tag{6.1.4}$$

we can expand d_{14} and $d_{14\beta}$ and appropriate versions of u and u_β in powers of s. Letting

$$v = \operatorname{sgn}[s^{1/2}\beta + s(1+\beta^2)/2]$$

we have

$$d_{14} = s^{-1}\left[\beta^2 - \frac{\beta^4}{3} + \frac{\beta^6}{15} - \frac{\beta^8}{105} + \ldots\right] - c_1^{-1} v\left[s^{1/2}\beta + s(\beta^2+1)/2\right] \tag{6.1.5}$$

$$(6.1.6) \qquad d_{14\beta} = s^{-1}\left[2\beta - \frac{4}{3}\beta^3 + \frac{6}{15}\beta^5 - \frac{8}{105}\beta^7 + \ldots\right] - c_1^{-1}v\left[s^{1/2} + s\beta\right].$$

Note that for small s,

$$(6.1.7) \quad s^{1/2}\beta + s(\beta^2+1)/2 = s^2\left[\frac{1-4a_1}{8}\right] + s^3\left[\frac{a_1-2a_2}{4}\right] + s^4\left[\frac{a_1^2+2a_2-4a_3}{8}\right] + \ldots$$

and hence,

$$(6.1.8) \qquad v = \operatorname{sgn}(1-4a_1)$$

Now we have

$$\begin{aligned} d_{14} &= \frac{1}{4} + s\left[\frac{a_1}{2} - \frac{1}{48}\right] + s^2\left[\frac{a_2}{2} + \frac{a_1^2}{4} - \frac{a_1}{12} + \frac{1}{960} + c_1^{-1}v\left(\frac{a_1}{2} - \frac{1}{8}\right)\right] \\ (6.1.9) \qquad &+ s^3\left[\frac{a_3}{2} + \frac{a_1a_2}{2} - \frac{a_2}{12} - \frac{1}{8}a_1^2 + \frac{a_1}{160} - \frac{1}{26880} + c_1^{-1}v\left(\frac{2a_2-a_1}{4}\right)\right] + \ldots \end{aligned}$$

$$\begin{aligned} d_{14\beta} &= -s^{-1/2}\Big\{1 + s\left[a_1 - \frac{1}{6} + c_1^{-1}v\right] + s^2\left[a_2 - \frac{1}{2}a_1 + \frac{1}{80} - \frac{c_1^{-1}v}{2}\right] \\ (6.1.10) \qquad &+ s^3\left[a_3 - \frac{1}{2}a_2 - \frac{1}{2}a_1^2 + \frac{1}{16}a_1 - \frac{1}{1680} - c_1^{-1}v\frac{a_1}{2}\right]\Big\} \end{aligned}$$

At this point we introduce a first approximation to the optimal solution of the heat equation which will match d_{14} and $d_{14\beta}$ on the inner and outer boundaries determined by appropriate terms $(a_{1i}, a_{2i}, \ldots)$ and $(a_{1o}, a_{2o}, \ldots)$ with $a_{1i} > 1/4 > a_{1o}$ so that $v_i = \operatorname{sgn}(1-4a_{1i}) < 0$ and $v_o = \operatorname{sgn}(1-4a_{1o}) > 0$. Let

$$(6.1.11) \qquad u_1 = r_1 s^{-1/2} J(\beta) + r_2$$

with

$$(6.1.12) \qquad u_{1\beta} = r_1 s^{-1/2} J'(\beta).$$

Expanding, we have

$$\begin{aligned} u_1 &= \left[r_2 - \frac{1}{2}r_1\right] + r_1 s\left[\frac{-a_1}{2} + \frac{1}{24}\right] + r_1 s^2\left[\frac{-a_2}{2} + \frac{a_1}{8} - \frac{1}{480}\right] \\ (6.1.13) \qquad &+ r_1 s^3\left[\frac{-a_3}{2} + \frac{a_2}{8} + \frac{a_1^2}{8} - \frac{a_1}{96} + \frac{1}{13440}\right] + \ldots \end{aligned}$$

$$u_{1\beta} = r_1 s^{-1/2}\left\{1 - s\cdot\frac{1}{4} + s^2\left[\frac{-a_1}{2} + \frac{1}{48}\right] + s^3\left[\frac{-a_2}{2} - \frac{a_1^2}{4} + \frac{a_1}{12} - \frac{1}{960}\right] + \dots\right\} \tag{6.1.14}$$

Matching coefficients alternately with $d_{14\beta}$ and d_{14}, we have

$$r_1 = -1 \tag{6.1.15}$$

and

$$r_2 = -1/4 \tag{6.1.16}$$

We may also proceed further after noting that

$$\begin{aligned} d_{14} - u_1 = {} & s\left[\frac{1}{48}\right] + s^2\left[\frac{a_1^2}{4} + a_1\left(\frac{1}{24} + \frac{c_1^{-1}v}{2}\right) - \left(\frac{1}{960} + \frac{c_1^{-1}v}{8}\right)\right] \\ & + s^3\left[\frac{a_1 a_2}{2} + a_2\left(\frac{1}{24} + \frac{c_1^{-1}v}{2}\right) - a_1\left(\frac{1}{240} + \frac{c_1^{-1}v}{4}\right) + \frac{1}{26880}\right] + \dots \end{aligned} \tag{6.1.17}$$

and

$$\begin{aligned} d_{14\beta} - u_{1\beta} = -s^{-1/2}\Big\{ & s\left[a_1 + \frac{1}{12} + c_1^{-1}v\right] + s^2\left[a_2 - \frac{1}{120} - \frac{c_1^{-1}v}{2}\right] \\ & + s^3\left[a_3 - \frac{a_1^2}{4} + a_1\left(-\frac{1}{48} - \frac{c_1^{-1}v}{2}\right) + \frac{1}{2240}\right] + \dots\Big\} \end{aligned} \tag{6.1.18}$$

By adjoining to u_1 the solution (of the heat equation)

$$u_2 = -r_3 s^{1/2}\beta + r_4 s(1+\beta^2) \tag{6.1.19}$$

with

$$u_{2\beta} = -r_3 s^{1/2} + 2r_4 s\beta \tag{6.1.20}$$

we match the coefficients of $s^{1/2}$ and s in $d_{14\beta} - u_{1\beta} - u_{2\beta}$ and $d_{14} - u_1 - u_2$.

We have

$$u_2 = s\left[\frac{r_3}{2} + r_4\right] + s^2\left[\frac{a_1 r_3}{2} + \frac{r_4}{4}\right] + s^3\left[\frac{a_2 r_3}{2} + \frac{a_1 r_4}{2}\right] + \dots \tag{6.1.21}$$

$$u_{2\beta} = -s^{-1/2}\left\{sr_3 + s^2 r_4 + s^3 a_1 r_4 + \dots\right\} \tag{6.1.22}$$

This leads to

$$r_3 = a_1 + \frac{1}{12} + c_1^{-1} v \tag{6.1.23}$$

and

$$\frac{r_3}{2} + r_4 = \frac{1}{48} \tag{6.1.24}$$

But the first equation represents two equations depending on v, the sign of $(1 - 4a_1)$. It stands for

$$r_3 = a_{1i} + \frac{1}{12} - c_1^{-1} \tag{6.1.25}$$

$$r_3 = a_{1o} + \frac{1}{12} + c_1^{-1} \tag{6.1.26}$$

or

$$a_{1o} = a_{1i} - 2c_1^{-1} \tag{6.1.27}$$

At this point we have expressed a_{10}, r_3 and

$$r_4 = \frac{1}{48} - \frac{r_3}{2} = \frac{-a_{1i}}{2} - \frac{1}{48} + \frac{c_1^{-1}}{2} \tag{6.1.28}$$

in terms of an undetermined parameter $a_{1i} > 1/4$. Now we introduce

$$u_3 = -r_5 s^{3/2}\left(\beta + \frac{\beta^3}{3}\right) + r_6 s^2\left(1 + 2\beta^2 + \frac{\beta^4}{3}\right) \tag{6.1.29}$$

with

$$u_{3\beta} = -r_5 s^{3/2}(1 + \beta^2) + 4r_6 s^2\left(\beta + \frac{\beta^3}{3}\right) \tag{6.1.30}$$

Expanding along the boundaries we have

$$u_3 = s^2\left[\frac{r_5}{2} + r_6\right] + s^3\left[\frac{a_1 r_5}{2} + \frac{r_5}{24} + \frac{r_6}{2}\right] + \ldots \tag{6.1.31}$$

and

$$u_{3\beta} = -s^{-1/2}\left\{r_5 s^2 + s^3\left[\frac{r_5}{4} + 2r_6\right] + \ldots\right\} \tag{6.1.32}$$

Setting the coefficients of s^2 and $s^{3/2}$ in $d_{14} - u_1 - u_2 - u_3$ and $d_{14\beta} - u_{1\beta} - u_{2\beta} - u_{3\beta}$ equal to zero, we have

$$(6.1.33) \qquad \frac{a_1^2}{4} + a_1\left(\frac{1}{24} + \frac{c_1^{-1}v}{2}\right) - \left(\frac{1}{960} + \frac{c_1^{-1}v}{8}\right) - \left(\frac{a_1 r_3}{2} + \frac{r_4}{4}\right) = \frac{r_5}{2} + r_6$$

and

$$(6.1.34) \qquad a_2 - \frac{1}{120} - \frac{c_1^{-1}v}{2} - r_4 = r_5 \ .$$

Considering that a_1 and v can take on two distinct values in (6.1.33) and r_3, r_4, r_5 and r_6 are restricted to single values, it follows that

$$(6.1.35) \quad \frac{a_{1i}^2}{4} + a_{1i}\left(\frac{1}{24} - \frac{c_1^{-1}}{2}\right) + \frac{c_1^{-1}}{8} - a_{1i}\frac{r_3}{2} = \frac{a_{1o}^2}{4} + a_{1o}\left(\frac{1}{24} + \frac{c_1^{-1}}{2}\right) - \frac{c_1^{-1}}{8} - a_{1o}\frac{r_3}{2} \ .$$

Since $a_{1i} - a_{1o} = 2c_1^{-1}$ and $(a_{1i}^2 - a_{1o}^2) = (a_{1i} + a_{1o})2c_1^{-1} = 4(a_{1i} - c_1^{-1})c_1^{-1}$, it follows that

$$(6.1.36) \qquad r_3 = 1/3 \ ,$$

and

$$(6.1.37) \qquad r_4 = -\frac{7}{48} \ ,$$

$$(6.1.38) \qquad a_{1i} = \frac{1}{4} + c_1^{-1} \ ,$$

and

$$(6.1.39) \qquad a_{1o} = \frac{1}{4} - c_1^{-1} \ .$$

From (6.1.34) it follows that

$$(6.1.40) \qquad a_{2i} - a_{2o} = -c_1^{-1} \ .$$

Now we introduce

$$(6.1.41) \qquad u_4 = -r_7 s^{5/2}(\beta + \frac{2}{3}\beta^3 + \frac{1}{15}\beta^5) + r_8 s^3(1 + 3\beta^2 + \beta^4 + \frac{1}{15}\beta^6)$$

and

(6.1.42) $$u_{4\beta} = -r_7 s^{5/2}(1 + 2\beta^2 + \frac{1}{3}\beta^4) + r_8 s^3(6\beta + 4\beta^3 + \frac{2}{5}\beta^5) .$$

Expanding along the boundaries we have

(6.1.43) $$u_4 = s^3(\frac{r_7}{2} + r_8) + \dots$$

and

(6.1.44) $$u_{4\beta} = -s^{-1/2}\{r_7 s^3 + \dots\} .$$

Matching coefficients of s^3 and $s^{5/2}$ in $d_{14} - u_1 - u_2 - u_3 - u_4$ and its derivative with respect to β, we have

(6.1.45) $$\begin{aligned} &\frac{a_1 a_2}{2} + a_2\left(\frac{1}{24} + \frac{c_1^{-1}v}{2}\right) + a_1\left(-\frac{1}{240} - \frac{c_1^{-1}v}{4}\right) - \frac{1}{26880} - \left(\frac{a_2 r_3}{2} + \frac{a_1 r_4}{2}\right) \\ &- \left(\frac{a_1 r_5}{2} + \frac{r_5}{24} + \frac{r_6}{2}\right) = \frac{1}{2}r_7 + r_8 \end{aligned}$$

and

(6.1.46) $$a_3 - \frac{a_1^2}{4} + a_1\left(-\frac{1}{48} - \frac{c_1^{-1}v}{2}\right) + \frac{1}{2240} - a_1 r_4 - \left(\frac{r_5}{4} + 2r_6\right) = r_7 .$$

Since the r_j do not depend on whether we use a_{ji} or a_{jo}, the equation (6.1.45) implies, with some algebraic labor,

(6.1.47) $$r_5 = \frac{31}{80}$$

which combined with (6.1.33) and (6.1.34) yield

(6.1.48) $$a_{2i} = \frac{1}{4} - \frac{c_1^{-1}}{2}$$

(6.1.49) $$a_{20} = \frac{1}{4} + \frac{c_1^{-1}}{2}$$

and

(6.1.50) $$r_6 = -\frac{167}{960} - \frac{c_1^{-2}}{4} .$$

Applying (6.1.46) to a_{ji} and a_{jo}, it follows that

$$a_{3i} = a_{3o} - \frac{1}{4}c_1^{-1} \tag{6.1.51}$$

and r_7 and r_8 can be evaluated in terms of a_{3o}. Thus far we have for the inner and outer boundaries

$$\tilde{z}_i = \tilde{y}_i - 1 = -\frac{s}{2}\left\{1 + \left(\frac{1}{4} + c_1^{-1}\right)s + \left(\frac{1}{4} - \frac{c_1^{-1}}{2}\right)s^2 + \dots\right\} \tag{6.1.52}$$

$$\tilde{z}_0 = \tilde{y}_0 - 1 = -\frac{s}{2}\left\{1 + \left(\frac{1}{4} - c_1^{-1}\right)s + \left(\frac{1}{4} + \frac{c_1^{-1}}{2}\right)s^2 + \dots\right\} \tag{6.1.53}$$

which compare with $\tilde{y}^2 + s = 1$ where

$$\tilde{y} = (1-s)^{-1/2} = 1 - \frac{s}{2}\left\{1 + \frac{1}{4}s + \frac{1}{8}s^2 + \frac{5}{64}s^3 + \dots\right\} \tag{6.1.54}$$

The approximation to the optimal risk given by $u = u_1 + u_2 + u_3 + \dots$ is

$$\begin{aligned} u = &-s^{-1/2}J(\beta) - \frac{1}{4} - \frac{1}{3}z - \frac{7}{48}(s+z^2) \\ &- \frac{31}{80}\left(sz + \frac{z^3}{3}\right) - \left(\frac{167}{960} + \frac{c_1^{-2}}{4}\right)\left(s^2 + 2sz^2 + \frac{z^4}{3}\right) + \dots \end{aligned} \tag{6.1.55}$$

Deriving further terms becomes straightforward, but tedious, and requires considerable care.

6.2 Problem 2, s small

In Problem 2, the boundary is also near $y = \pm 1$ when s is small, and once again we translate to $z = y - 1$. We have for large $|\alpha|$

$$\begin{aligned} G_{1e}(\alpha) &= 2[\phi(\alpha) - |\alpha|(1 - \phi(|\alpha|))] + |\alpha| \\ &\approx |\alpha| + \phi(\alpha)\left\{\frac{1}{\alpha^2} - \frac{3}{\alpha^4} + \dots\right\} \end{aligned} \tag{6.2.1}$$

Thus for small s and y close to one, $G_{1e}(\alpha)$ is very close to $|\alpha|$ and

$$\begin{aligned} d_{22}(y,s) &= c_2 s^{-1} + \min[1, s^{1/2}G_{1e}(\alpha)] \\ &\approx c_2 s^{-1} + \min(1,y) = 1 + z/2 + d_{23}(z,s) \end{aligned} \tag{6.2.2}$$

where

$$d_{23}(z,s) = c_2 s^{-1} - |z|/2 \; . \tag{6.2.3}$$

With the transformation

$$z^* = az, \;\; s^* = a^2 s, \;\; a = (2c_2)^{-1/3} \tag{6.2.4}$$

we have

$$d_{23}(z,s) = (c_2/4)^{1/3}[d_{24}(z^*,s^*) + s^{*-1}J'(z^* s^{*-1/2})] \tag{6.2.5}$$

where

$$d_{24}(z,s) = s^{-1}[1 - J'(\beta)] - s^{1/2}|\beta| \tag{6.2.6}$$

is symmetric in β, and therefore easier to handle than d_{14}.

We attack the asymptotic expansion for d_{24} near $s = 0$ and $z = 0$ by assuming

$$\beta = a_1 s^{3/2} + a_2 s^{9/2} + a_3 s^{15/2} + \ldots$$

in which case we can expand, for positive β

$$\begin{aligned} d_{24} &= s^2(a_1^2 - a_1) + s^5\left[2a_1a_2 - \frac{a_1^4}{3} - a_2\right] \\ &\quad + s^8\left[2a_1a_3 + a_2^2 - \frac{4}{3}a_1^3a_2 + \frac{1}{15}a_1^6 - a_3\right] + \ldots \end{aligned} \tag{6.2.8}$$

$$\begin{aligned} d_{24\beta} &= s^{1/2}\left\{(2a_1 - 1) + s^3\left(2a_2 - \frac{4}{3}a_1^3\right)\right. \\ &\quad \left. + s^6\left(2a_3 - 4a_1^2a_2 + \frac{2}{5}a_1^5\right) + \ldots\right\} \end{aligned} \tag{6.2.9}$$

Let

$$\begin{aligned} u_1 = r_1 s^2 H_{4e}(\beta) &= r_1 s^2\{1 + 2\beta^2 + \beta^4/3\} \\ &= r_1 s^2\left\{1 + s^3(2a_1^2) + s^6(4a_1a_2 + a_1^4/3) + s^9(4a_1a_3\right. \\ &\qquad \left. + 2a_2^2 + 4a_1^3a_2/3) + \ldots\right\} \end{aligned} \tag{6.2.10}$$

$$u_{1\beta} = r_1 s^2 \left[4\beta + \frac{4}{3}\beta^3\right]$$

$$= r_1 s^{7/2}\left\{4a_1 + s^3[4a_2 + 4a_1^3/3] + s^6[4a_3 + 4a_1^2 a_2] + \dots\right\} \tag{6.2.11}$$

To match coefficients we need $2a_1 - 1 = 0$ or

$$a_1 = 1/2 \tag{6.2.12}$$

and $a_1^2 - a_1 = r_1$ or

$$r_1 = -1/4. \tag{6.2.13}$$

But then

$$d_{24} - u_1 = s^5\left[\frac{5}{48}\right] + s^8\left[a_2^2 - \frac{1}{6}a_2 + \frac{1}{960} + \frac{a_2}{2} + \frac{1}{192}\right] + \dots \tag{6.2.14}$$

$$d_{24\beta} - u_{1\beta} = s^{1/2}\left\{s^3\left[2a_2 - \frac{1}{6} + \frac{1}{2}\right] + s^6\left[2a_3 + \frac{1}{80} + \frac{a_1^3}{3}\right] + \dots\right\} \tag{6.2.15}$$

which suggest $2a_2 - 1/6 + 1/2 = 0$ or

$$a_2 = -1/6\ . \tag{6.2.16}$$

Since the coefficient of s^5 in $d_{24} - u_1$ is $5/48$ we take

$$u_2 = \frac{5}{48}s^5 H_{10e}(\beta) = \frac{5s^5}{48}\left[1 + 5\beta^2 + \frac{10}{3}\beta^4 + \dots\right]$$

$$= s^5\frac{5}{48} + s^8\frac{5a_1^2}{48} + \dots = s^5\frac{1}{48} + s^8\frac{5}{192} + \dots \tag{6.2.17}$$

and

$$u_{2\beta} = \frac{5s^5}{48}\left[10\beta + \frac{40}{3}\beta^3 + \dots\right]$$

$$= s^{1/2}\left\{s^6\left[\frac{50}{48}a_1\right] + s^9\left[\frac{50}{48}a_2 + \frac{25}{18}a_1^3\right] + \dots\right\}$$

$$= s^{1/2}\left\{s^6\left(\frac{25}{48}\right) + s^9(0) + \dots\right\} \tag{6.2.18}$$

Now the coefficient of $s^{1/2}s^6$ in $d_{24\beta} - u_{1\beta}$ is $2a_3 + \dfrac{13}{240} = \dfrac{25}{48}$ and

$$a_3 = 7/30\ . \tag{6.2.19}$$

We can continue in this manner, taking advantage of the symmetry of the problem. In the meantime we have obtained

$$|\tilde{z}| = \frac{1}{2}s^2 - \frac{1}{6}s^5 + \frac{7}{30}s^8 + \dots \tag{6.2.20}$$

for the inner and outer boundaries near $(y,s) = (1,0)$. The risk function can be approximated by $u_1 + u_2$. Note that this expansion applies to d_{24} where the variables z and s refer to az and $a^2 s$ in the original coordinates with $a = (2c_2)^{-1/3}$, i.e.,

$$|\tilde{y} - 1| = \frac{a^3}{2}s^2 - \frac{1}{6}a^9 s^5 + \frac{7}{30}a^{15}s^8 + \dots.$$

6.3 Problem 1, large s.

For c large enough, the continuation region is bounded. But for $c < \tilde{c}_{1o}$ it becomes unbounded and it is desirable to find an asymptotic expansion for the boundary and risk function when s is large (which corresponds to a great deal of prior uncertainty about μ). From Proposition 6 in Section 5 we know that for $c < c_{13}$, and $s > 1$, that the optimal boundary will have $|\tilde{\alpha}| \geq \alpha_0$.

This problem, where

$$d_{12}(y,s) = c_1 s^{-1} + 1 \qquad \text{for} \qquad s \geq 1$$

represents a somewhat unusual case for large s. For a stopping risk $c_1 s^{-1} + 1$ for all s, the optimal strategy is obviously that of never continuing. It is only the behavior for $s \leq 1$ that leads to the possibility of continuing when s is large.

In contrast with this problem, consider the canonical version of that related to deciding the sign of the mean drift of a Wiener process, the discrete time version of which is to decide the sign of the mean of a normal distribution. In that problem where

$$d = s^{-1} + s^{1/2} G_{1e}(ys^{-1/2}),$$

the main component of the optimal Bayes risk for large s and moderate α is a multiple of $s^{1/2}\phi(ys^{-1/2})$. There is another component which is relatively small for large s and moderate $\alpha = ys^{-1/2}$. But for large α, where the boundary is located, the first term becomes negligible and it is the second term which controls the location of the boundary.

This fact has statistical significance. On one hand the solution of the sequential problem is related to the backward induction of dynamic programming, and depends on the structure of the losses when s is small or, after much sampling, has reduced the uncertainty about the unknown mean μ. The nature of the losses after considerble sampling has an effect on our expected loss and the optimal stopping rule. On the other hand it has only a minor effect on an early decision of whether evidence is overwhelming enough to stop. In brief, in statistical decision problem, early decisions are typically insensitive to the consequences of potential late decisions.

Our current problem lacks this property. One consequence is that our asymptotic expansions, after the main terms, will depend on some constant which depends on the nature of the stopping risk for $0 < s < 1$. That constant is not available now. In the problem of deciding the sign of the mean, bounds on the corresponding constant were derived by Bickel and Yahav (1967) and Mallik and Yao (1984).

First we drop the 1 from d_{12} and so we have $d_{15} = c_1 s^{-1}$. Now let us represent a solution of the heat equation with the form

$$(6.3.1) \qquad u = r_0 s^{-1/2}\phi(\alpha) + \int_{-\infty}^{\infty} \phi(w)h(y + ws^{1/2})dw = r_0 v_0 + v$$

while equations (4.54-4.56) would be more rigorous. Here, our first approximation to $h(y)$, giving rise to v_1, is

$$(6.3.2) \qquad h_1(y) = r_1 y^{-2}\log y^2$$

for $y^2 > 1$ and 0 elsewhere, and we assume that along the boundary, for large s

$$(6.3.3) \qquad \log s = \alpha^2 + 2a_{-1}\log\alpha^2 + 2\left(a_0 + a_1\alpha^{-2} + a_2\alpha^{-4} + \ldots\right)$$

Then

$$(6.3.4) \qquad v_0 = s^{-1/2}\phi(\alpha) = s^{-1}(\alpha^2)^{a_{-1}}\frac{e^{a_0}}{\sqrt{2\pi}}e^{a_1\alpha^{-2}+a_2\alpha^{-4}+\ldots},$$

$$(6.3.5) \qquad v_0 \approx s^{-1}(\log s)^{a_{-1}}e^{a_0}\sqrt{2\pi},$$

and

$$(6.3.6) \qquad v_{0y} = -s^{-1}\alpha\phi(\alpha) = -s^{-3/2}(\alpha^2)^{a_{-1}+1/2}\frac{e^{a_0}}{\sqrt{2\pi}}e^{a_1\alpha^{-2}+a_2\alpha^{-4}+\ldots}$$

$$v_{0y} \approx -s^{-3/2}(\log s)^{(a_{-1}+1/2)}e^{a_0}/\sqrt{2\pi} \tag{6.3.7}$$

In the meantime

$$\begin{aligned}
h_1'(y) &= -r_1 y^{-3}(2\log y^2 - 2)\\
h_1^{(2)}(y) &= r_1 y^{-4}(6\log y^2 - 10)\\
h_1^{(3)}(y) &= -r_1 y^{-5}(24\log y^2 - 52)\\
h_1^{(4)}(y) &= r_1 y^{-6}(120\log y^2 - 308)\\
h_1^{(5)}(y) &= -r_1 y^{-7}(720\log y^2 - 2088)
\end{aligned}$$

and so

$$\begin{aligned}
v_1 = r_1 s^{-1}\Big[\alpha^{-2}\log(s\alpha^2) &+ \frac{1}{2}\alpha^{-4}\Big[6\log(s\alpha^2) - 10\Big]\\
&+ \frac{1}{8}\alpha^{-6}\Big[120\log(s\alpha^2) - 308\Big] + \dots\Big\}
\end{aligned} \tag{6.3.8}$$

and

$$\begin{aligned}
v_{1y} = -r_1 s^{-3/2}\alpha^{-1}\Big\{\alpha^{-2}\Big[2\log(s\alpha^2) - 2\Big] &+ \frac{1}{2}\alpha^{-4}\Big[24\log(s\alpha^2) - 52\Big]\\
&+ \frac{1}{8}\alpha^{-6}\Big[720\log(s\alpha^2) - 2088\Big] + \dots\Big\}
\end{aligned} \tag{6.3.9}$$

We would like to have,

$$r_0 v_0 + v_1 \approx c_1 s^{-1} \tag{6.3.10}$$

and

$$r_0 v_{0y} + v_{1y} \approx 0\ . \tag{6.3.11}$$

Since $v_{1y} \approx -2r_1 s^{-3/2}(\log s)^{-1/2}$, (6.3.11) requires $a_{-1} + 1/2 = -1/2$ or

$$a_{-1} = -1 \tag{6.3.12}$$

and

$$r_0 e^{a_0}/\sqrt{2\pi} = -2r_1 \tag{6.3.13}$$

But then (6.3.10) requires

$$r_1 = c_1 \,. \tag{6.3.14}$$

Note that

$$\log(s\alpha^2) = \alpha^2 - \log\alpha^2 + 2a_0 + 2a_1\alpha^{-2} + 2\alpha_2\alpha^{-4} + \dots$$

and thus, applying (4.55) and (4.56)

$$\begin{aligned}
v_1 &= h_1(s^{1/2}\alpha) + \frac{s}{2}h_1^{(2)}(s^{1/2}\alpha) + \frac{s^2}{8}h_1^{(4)}(s^{(1/2}\alpha) + \dots \\
v_1 &= c_1 s^{-1}\Big\{\Big[1 + (-\log\alpha^2 + 2a_0)\alpha^{-2} + 2a_1\alpha^{-4} + 2a_2\alpha^{-6}\Big] \\
&\quad + \Big[3\alpha^{-2} + (-3\log\alpha^2 + 6a_0 - 5)\alpha^{-4} + 6a_1\alpha^{-6}\Big] \\
&\quad + \Big[15\alpha^{-4} + (-15\log\alpha^2 + 30a_0 - 77/2)\alpha^{-6}\Big] + 105\alpha^{-6} + \dots\Big\}
\end{aligned} \tag{6.3.15}$$

and

$$\begin{aligned}
v_{1y} &= h_1'(s^{1/2}\alpha) + \frac{s}{2}h_1^{(3)}(s^{1/2}) + \dots \\
v_{1y} &= -c_1 s^{-3/2}\alpha^{-1}\Big\{\Big[2 + (-2\log\alpha^2 + 4a_0 - 2)\alpha^{-2} + 4a_1\alpha^{-4} + 4a_2\alpha^{-6}\Big] \\
&\quad + \Big[12\alpha^{-2} + (-12\log\alpha^2 + 24a_0 - 26)\alpha^{-4} + 24a_1\alpha^{-6}\Big] \\
&\quad + \Big[90\alpha^{-4} + (-90\log\alpha^2 + 180a_0 - 261)\alpha^{-6}\Big] + 840\alpha^{-6} + \dots\Big\} \\
r_0 v_0 &= -2c_1 s^{-1}\left\{\alpha^{-2} + a_1\alpha^{-4} + \left(a_2 + \frac{a_1^2}{2}\right)\alpha^{-6} + \left(a_3 + a_1a_2 + \frac{a_1^3}{6}\right)\alpha^{-8} + \dots\right\} \\
r_0 v_{0y} &= 2c_1 s^{-3/2}\alpha^{-1}\left\{1 + a_1\alpha^{-2} + \left(a_2 + \frac{a_1^2}{2}\right)\alpha^{-4}\right. \\
&\quad \left. + \left(a_3 + a_1a_2 + \frac{a_1^3}{6}\right)\alpha^{-6} + \dots\right\}
\end{aligned} \tag{6.3.16}$$

Now

$$r_0 v_{0y} + v_{1y} \approx c_1 s^{-3/2}\Big\{\alpha^{-3}\Big[2a_1 + 2\log\alpha^2 - 4a_0 + 2 - 12\Big] + \dots\Big\} \tag{6.3.17}$$

and

$$r_0 v_0 + v_1 - c_1 s^{-1} \approx c_1 s^{-1}\Big\{\alpha^{-2}\Big[-2 - \log\alpha^2 + 2a_0 + 3\Big] + \dots\Big\} \,. \tag{6.3.18}$$

We let

$$h_2(y) = y^{-2}\left\{ r_{21} \log[\log y^2] + r_{22} \right\} \tag{6.3.19}$$

which gives rise to $v_2 = h_2 + sh_2^{(2)}/2 + \ldots$ and $v_{2y} = h_2' + sh_2^{(3)}/2 + \ldots$, or

$$\begin{aligned} v_2 &\approx s^{-1}\alpha^{-2}\left\{ r_{21} \log\left[\alpha^2 - \log\alpha^2 + 2a_0 + 2a_1\alpha^{-2} + \ldots\right] + r_{22} + \ldots \right\} \\ &= s^{-1}\alpha^{-2}\left\{ r_{21}\log\alpha^2 + r_{22} + r_{21}\alpha^{-2}\left(-\log\alpha^2 + 2a_0 + 2a_1\alpha^{-2} + \ldots \right) + \ldots \right\} \end{aligned}$$

$$v_2 \approx s^{-1}\alpha^{-2}\left\{ r_{21}\log\alpha^2 + r_{22} + r_{21}\alpha^{-2}\left(-\log\alpha^2 + 2a_0 + \ldots \right) + \ldots \right\} \tag{6.3.20}$$

and

$$v_{2y} = -s^{-3/2}\alpha^{-3}\left\{ 2r_{21}\log\alpha^2 + 2r_{22} + \ldots \right\} \tag{6.3.21}$$

from which it follows that

$$\begin{aligned} 2c_1 - 2r_{21} &= 0 \\ c_1(2a_1 - 4a_0 - 10) - 2r_{22} &= 0 \\ -c_1 + r_{21} &= 0 \\ c_1(-2 + 2a_0 + 3) + r_{22} &= 0 \end{aligned}$$

or

$$r_{21} = c_1 \tag{6.3.22}$$

$$r_{22} = -c_1(2a_0 + 1) = c_1(a_1 - 2a_0 - 5) \tag{6.3.23}$$

and

$$a_1 = 4\,. \tag{6.3.24}$$

Thus our current approximations involve an unknown parameter a_0. We have

$$\log s = \alpha^2 - 2\log\alpha^2 + 2a_0 + 8\alpha^{-2} + \ldots \tag{6.3.25}$$

and

$$u = c_1 s^{-1/2}\left\{-2\sqrt{2\pi}e^{-a_0}\phi(\alpha)\right.$$

(6.3.26) $$\left. + \int \phi\left(\frac{z-y}{\sqrt{s}}\right)z^{-2}\left[\log z^2 + \log(\log z^2) - (2a_0+1)\right]dz + \dots\right\}$$

The integral in the expression above does not converge because for small z , $\log(\log z^2)$ is not defined. For asymptotic purposes the value of the integrand in a bounded range of z values is unimportant and the result would be more meaningful if the integrand were replaced by zero over the range $|z| \leq 1$. Chernoff (1965a) has a detailed discussion, for the problem of deciding the sign of the mean, of why we should expect asymptotic approximations of the form

(6.3.27) $$u = s^{1/2}\left\{r_0\phi(\alpha) + r_0^* s^{-1/2}\alpha\phi(\alpha) + \int \phi(\frac{z-y}{\sqrt{s}})h(z)dz + \dots\right\}$$

and why they lead to asymptotic approximations. In the symmetric case $r^* = 0$.

While we still lack control of a_0, which is crucial in u, it plays a less prominent role in the expansion for the optimal boundary which can be written

(6.3.28) $$\tilde{\alpha}^2 = \log s + 2\log(\log s) - 2a_0 + (\log s)^{-1}[2\log(\log s) - 8 - 2a_0] + \dots$$

and for which the prominent terms do not depend on s. From one point of view this states that for large s one should stop and reject bioequivalence when the P value $2\Phi(-y/\sqrt{s})$ is less than the nominal significance level,

$$2\Phi(-\tilde{\alpha}) \approx \frac{2\phi(\tilde{\alpha})}{|\tilde{\alpha}|} \approx 2s^{-1/2}(\log s)^{-3/2}\frac{e^{a_0}}{\sqrt{2\pi}} .$$

Since a_0 is not known, this nominal significance level is as yet unspecified, but the order of magnitude is clear.

Finally, one word of caution. At this point it is not clear that the asymptotic expansion derived by considering more corrections will be as simple as (6.3.3). It may require replacing some of the coefficients $a_1, a_2, \dots$ by terms involving $\log\alpha^2$ or even $\log(\log\alpha^2)$.

6.4 Large s, Problem 2.

There is no need to elaborate here. This problem reduces to $d_{2y} = c_2 s^{-1} + 1$ for $s \geq 1$, and hence has the same expansion as in Section 6.3. The only difference is that c_1 is replaced by c_2 and we have a different unknown value for a_0.

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